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Moments of Ramanujan's Generalized Elliptic Integrals and Extensions of Catalan's Constant

D. Borwein,^{*} J.M. Borwein,[†] M.L. Glasser [‡]and J.G Wan[§]

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Abstract

We undertake a thorough investigation of the moments of Ramanujan's alternative elliptic integrals and of related hypergeometric functions. Along the way we are able to give some surprising closed forms for Catalan-related constants and various new hypergeometric identities.

Key words: elliptic integrals, hypergeometric functions, moments, Catalan's constant.

1 Introduction and background

As in [7, pp. 178-179], for $0 \leq s < 1/2$ and $0 \leq k \leq 1$, let

$$K^s(k) := \frac{\pi}{2} {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| k^2 \right) \quad (1)$$

and

$$E^s(k) := \frac{\pi}{2} {}_2F_1 \left(\begin{matrix} -\frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| k^2 \right). \quad (2)$$

We use the standard notation for hypergeometric functions, namely

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

^{*}Department of Mathematics, University of Western Ontario, London ON Canada.

[†]Corresponding author, CARMA, University of Newcastle, NSW 2308 Australia. Email: jonathan.borwein@newcastle.edu.au.

[‡]Department of Physics, Clarkson University Potsdam, NY 136-99-5820 (USA)

[§]CARMA, University of Newcastle, NSW 2308 Australia.

and its analytic continuation, where $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ is the rising factorial or *Pochhammer symbol*; likewise,

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| z \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{z^n}{n!}.$$

One of the key early results, due to Gauss (1812), is the closed form

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (3)$$

when $\operatorname{Re}(c-a-b) > 0$.

We are interested in the moments given by

$$K_n = K_{n,s} := \int_0^1 k^n K^s(k) dk, \quad E_n = E_{n,s} := \int_0^1 k^n E^s(k) dk. \quad (4)$$

for both integer and real values of n . We immediately note that $K^s = K^{(-s)}$. Also, Euler's transform [3, Eqn. (2.2.7)] and a contiguous relation yield

$$E^{(-s)} = \frac{4s(1-k^2)}{2s-1} K^s + \frac{2s+1}{2s-1} E^s.$$

The corresponding integral form of K^s may be obtained by expanding $(1-k^2t)^{s-1/2}$ and using the identity $\Gamma(1/2-s)\Gamma(1/2+s) = \pi/\cos(\pi s)$:

$$K^s(k) = \frac{\cos \pi s}{2} \int_0^1 \frac{t^{s-1/2}}{(1-t)^{1/2+s}(1-k^2t)^{1/2-s}} dt \quad (5)$$

$$= \cos(\pi s) \int_0^{\pi/2} \frac{\tan^{2s}(\theta)}{(1-k^2 \sin^2 \theta)^{1/2-s}} d\theta. \quad (6)$$

The latter has the nice feature of looking like the cleanest classical definition when $s = 0$. These and many more forms for K^s, E^s can be obtained from <http://dlmf.nist.gov/15.6>. There are four values for which these integrals are truly special:

$$s \in \Omega := \left\{ 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3} \right\},$$

that is, when $\cos^2(\pi s)$ is rational.

These are Ramanujan's alternative elliptic integrals as displayed in [13] and first decoded in [7]. A comprehensive study is given in [5] (see also [11] and [2]). These four cases are all produce modular functions [7, §5.5] and study is currently experiencing a renewal of interest, especially regarding related elliptic series for $1/\pi$ ([6], [7, §5.5] and [8]).

1.1 Reciprocal series for π

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910 [6, 7]. The most famous, with $s = 1/4$ is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}. \quad (7)$$

Each term of (7) adds eight correct digits. Gosper used (7) for the computation of a then-record 17 million digits of π in 1985 — thereby completing the first proof of (7) [7, Ch. 3]. Shortly thereafter, David and Gregory Chudnovsky found the following variant, which uses $s = 1/3$ and lies in the quadratic number field $\mathbb{Q}(\sqrt{-163})$ rather than $\mathbb{Q}(\sqrt{58})$:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}. \quad (8)$$

Each term of (8) adds 14 correct digits. The brothers used this formula several times, culminating in a 1994 calculation of π to over four billion decimal digits. Their extraordinary story was told in a prizewinning *New Yorker* article by Richard Preston. Remarkably, (8) was used again in late 2009 for the then-record computation of π to 2.7 trillion places. In consequence, Fabrice Bellard has provided access to two trillion-digit integers whose ratio is bizarrely close to π . A striking recent series due to Yao, see [16], is

$$\frac{1}{\pi} = \frac{\sqrt{15}}{18} \sum_{n=0}^{\infty} \frac{\sum_{k=0}^n \binom{n}{k}^4 (4n+1)}{36^n}. \quad (9)$$

1.2 Classical results

The coupling equation between E^s and K^s is given in [7, p. 178] and can be derived from the generalized hypergeometric differential equation (see <http://dlmf.nist.gov/15.10>). It is

$$E^s = (1 - k^2) K^s + \frac{k(1 - k^2)}{1 + 2s} \frac{d}{dk} K^s. \quad (10)$$

Integrating this by parts leads to

$$K_{2,s} = \frac{(1 + 2s) E_{0,s} - 2s K_{0,s}}{2 - 2s}. \quad (11)$$

In the same fashion, multiplying by k^n before integrating the coupling provides a recursion for $K_{n+2,s}$:

$$K_{n+2,s} = \frac{(n - 2s) K_{n,s} + (1 + 2s) E_{n,s}}{n + 2(1 - s)}. \quad (12)$$

We also consider the *complementary* integrals:

$$K'^s(k) := K^s(\sqrt{1 - k^2}) \quad \text{and} \quad E'^s(k) := E^s(\sqrt{1 - k^2}).$$

The four integrals then satisfy a version of *Legendre's identity*,

$$E^s K'^s + K^s E'^s - K^s K'^s = \frac{\pi}{2} \frac{\cos \pi s}{1 + 2s} \quad (13)$$

for all $0 \leq k \leq 1$.

In [7, pp. 198-99] the moments are determined for the classical case of $s = 0$ which give the original complete elliptic integrals K and E . These are linked by the equations (see [7, p. 9])

$$E = (1 - k^2) K + k(1 - k^2) \frac{dK}{dk}, \quad (14)$$

which is (10) with $s = 0$ and

$$E = K + k \frac{dE}{dk}, \quad (15)$$

from which we derive the following recursions:

Theorem 1 ($s = 0$) For $n = 0, 1, 2, \dots$

$$(a) K_{n+2} = \frac{nK_n + E_n}{n+2} \quad \text{and} \quad (b) E_n = \frac{K_n + 1}{n+2}. \quad (16)$$

The recursion holds for real n . Moreover,

$$K_0 = 2G, \quad K_1 = 1, \quad (17)$$

$$E_0 = G + \frac{1}{2}, \quad E_1 = \frac{2}{3}. \quad (18)$$

Here

$$G := \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} = L_{-4}(2)$$

is *Catalan's* constant whose irrationality is still not proven. This ignorance is part of our motivation for the current study. Indeed [1] uses this moment as a definition of G !

The current record for computation of G is 31.026 billion decimal digits in 2009. Computations often use the following central binomial formula due to Ramanujan [7, last formula] or its recent generalizations [10]:

$$\frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n} (2n+1)^2} + \frac{\pi}{8} \log(2 + \sqrt{3}) = G. \quad (19)$$

Early in 2011, a string of base-4096 digits of Catalan's constant beginning at position 10 trillion was computed on an IBM *Blue Gene/P* machine as part of a suite of similar computations [4]. The resulting confirmed base-8 digit string is

34705053774777051122613371620125257327217324522

(each quadruplet of base-8 digits corresponds to one base-4096 digit).

There are various ways to obtain the initial values, and one may also profitably study fractional moments, see below and [1].

2 Basic results

We commence in this section with various fundamental representations and evaluations. Then in section three we provide a generalization of Catalan's constant arising as the expectation of K^s . In section four we consider related contour integrals. Finally, in section five we look at negative and fractional moments.

2.1 Hypergeometric closed forms

A concise closed form for the moments is

Theorem 2 (Hypergeometric forms) For $0 \leq s < \frac{1}{2}$ we have

$$K_{n,s} = \frac{\pi}{2(n+1)} {}_3F_2 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix} \middle| 1 \right), \quad (20)$$

$$E_{n,s} = \frac{\pi}{2(n+1)} {}_3F_2 \left(\begin{matrix} -\frac{1}{2} - s, \frac{1}{2} + s, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix} \middle| 1 \right). \quad (21)$$

These hold in the limit for $s = \frac{1}{2}$.

Proof. To establish (20) and (21), we begin with

$$\begin{aligned} \int_0^1 x^{u-1}(1-x)^{v-1} {}_2F_1 \left(\begin{matrix} a, 1-a \\ b \end{matrix} \middle| x \right) dx &= \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(b)_n n!} \int_0^1 x^{n+u-1}(1-x)^{v-1} dx \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n (u)_n \Gamma(u)\Gamma(v)}{(b)_n (u+v)_n n! \Gamma(u+v)} \\ &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} {}_3F_2 \left(\begin{matrix} a, 1-a, u \\ b, u+v \end{matrix} \middle| 1 \right). \end{aligned} \quad (22)$$

Similarly,

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} {}_3F_2 \left(\begin{matrix} a, -a, u \\ b, u+v \end{matrix} \middle| 1 \right) = \int_0^1 x^{u-1}(1-x)^{v-1} {}_2F_1 \left(\begin{matrix} a, -a \\ b \end{matrix} \middle| x \right) dx.$$

By applying these to (1) and (2) we immediately get (20) and (21). \square

As long as $0 < s < 1/2$, the first series (20) is *Saalschützian* [14]. That is, the denominator parameters add to one more than those in the numerator, but is not well poised, and can be reduced to Gamma functions only for $n = \pm 1$ (with $n = -1$ a pole) since then it reduces to a ${}_2F_1$. The second (21) is not even Saalschützian, although it is nearly well poised (whose definition [14] we do not need) and also can be reduced to Gamma functions for $n = \pm 1$. Thus, for $|s| < 1/2$ we find

$$K_{1,s} = \frac{\cos \pi s}{1 - 4s^2}, \quad E_{1,s} = \frac{2}{2s + 3} \frac{\cos \pi s}{1 - 4s^2}. \quad (23)$$

In general we obtain:

Theorem 3 (Odd moments of K^s) For odd integers $2m + 1$ and $m = 0, 1, 2, \dots$,

$$K_{2m+1,s} = \frac{\cos \pi s m!^2}{4 \Gamma\left(\frac{3}{2} - s + m\right) \Gamma\left(\frac{3}{2} + s + m\right)} \sum_{k=0}^m \frac{\Gamma\left(\frac{1}{2} - s + k\right) \Gamma\left(\frac{1}{2} + s + k\right)}{k!^2}. \quad (24)$$

Proof. In terms of the *Legendre function*,

$${}_2F_1\left(a, 1-a \middle| z\right) =: P_{-a}(1-2z),$$

where

$$y = P_\nu(x) = {}_2F_1\left(-\nu, \nu+1 \middle| \frac{1-x}{2}\right)$$

is a solution of the differential equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \nu(\nu+1)y = 0.$$

In consequence we may deduce that

$$\begin{aligned} {}_2F_1\left(a, 1-a \middle| z\right) &= \frac{\sin \pi a}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{k!^2} (1-z)^k \times \\ &\quad \{2\Psi(1+k) - \Psi(a+k) - \Psi(1-a+k) - \log(1-z)\}, \end{aligned} \quad (25)$$

where

$$\Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt$$

is the *digamma* function, using [12, p. 44, first formula ($b = 1 - a$)].

Now, by integrating the series (25) term-by-term and applying representation (22), we have

$$\begin{aligned} {}_3F_2\left(a, 1-a, n \middle| 1\right) &= n \int_0^1 z^{n-1} {}_2F_1\left(a, 1-a \middle| z\right) dz \\ &= \frac{n! \sin \pi a}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{k!(k+n)!} \times \\ &\quad \{\Psi(1+k) + \Psi(n+1+k) - \Psi(a+k) - \Psi(1-a+k)\}. \end{aligned}$$

We note in passing that this offers an apparently new approach for summing this class of hypergeometric series; we exploit (22) again in section 5.4.

Thence, for example, by creative telescoping, one finds for any positive integer n that

$${}_3F_2\left(a, 1-a, n \middle| 1\right) = \frac{\Gamma(n) \Gamma(1+n)}{\Gamma(a+n) \Gamma(1-a+n)} \sum_{k=0}^{n-1} \frac{(a)_k (1-a)_k}{k!^2}. \quad (26)$$

Now, with $n = m + 1$ in (26) we conclude the proof of Theorem 3. \square

Similarly,

$${}_2F_1\left(\begin{matrix} a, -a \\ 1 \end{matrix} \middle| z\right) = \frac{\sin(\pi a)}{\pi a} \left\{ 1 - a^2 \sum_{k=0}^{\infty} \frac{(a+1)_k (1-a)_k}{k!(k+1)!} (1-z)^{k+1} \times \right. \\ \left. [\Psi(a+1+k) + \Psi(1-a+k) - \Psi(k+1) - \Psi(k+2) + \ln(1-z)] \right\}.$$

For $m = 0$, Theorem 3 reduces to the evaluation given in (23). In general, it gives $\cos(\pi s)$ times a rational function. An equivalent, rather pretty, partial fraction decomposition is

$$K_{2m+1,s} = \frac{\cos \pi s}{2} \sum_{k=0}^m \frac{m!^2}{(m-k)!(m+k+1)!} \left(\frac{1}{2k+1-2s} + \frac{1}{2k+1+2s} \right). \quad (27)$$

This can easily be confirmed inductively, using say (76).

For $s = 0$ this result originates with Ramanujan. Adamchik [1] reprises its substantial history and extensions which include a formula due independently to Bailey and Hodgkinson in 1931 and which subsumes (26). A special case of Bailey's formula is

$${}_3F_2\left(\begin{matrix} a, b, c+1 \\ a+b+n \end{matrix} \middle| 1\right) = \frac{\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+n)\Gamma(b+n)} \sum_k^{n-1} \frac{(a)_k (b)_k}{(c)_k (1)_k}. \quad (28)$$

Example 1 (Digamma consequences) For $0 < a < 1/2$, consequences are neatly given using:

$$\gamma(\nu) := \frac{1}{2} \left[\Psi\left(\frac{\nu+1}{2}\right) - \Psi\left(\frac{\nu}{2}\right) \right],$$

for which

$$\begin{aligned} \gamma\left(\frac{1}{2}\right) &= \frac{\pi}{2}, & \gamma\left(\frac{1}{4}\right) &= \frac{\pi}{\sqrt{2}} - \sqrt{2} \log(\sqrt{2}-1), \\ \gamma\left(\frac{1}{3}\right) &= \frac{\pi}{\sqrt{3}} + \log 2, & \gamma\left(\frac{1}{6}\right) &= \pi + \sqrt{3} \log(2 + \sqrt{3}). \end{aligned}$$

More generally,

$$\sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{\left(\frac{3}{2}\right)_k k!} \left[\Psi(k+1) + \Psi\left(k + \frac{3}{2}\right) - \Psi(k+a) - \Psi(k+1-a) \right] = \frac{2\gamma(a) - \pi \csc(\pi a)}{1-2a}.$$

This in turn gives

$${}_3F_2\left(\begin{matrix} a, 1-a, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1\right) = \frac{2 \sin(\pi a)}{\pi(1-2a)} \gamma(a) - \frac{1}{1-2a}. \quad (29)$$

Taking the limit as $a \rightarrow 1/2$ in (29) gives two useful specializations:

$$(a) \quad {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1\right) = \frac{4G}{\pi} \quad (30)$$

$$(b) \quad \Psi'\left(\frac{1}{4}\right) = \pi^2 + 8G, \quad (31)$$

with (30) being known but far from obvious. \diamond

Example 2 (Odd moments of E^s) The corresponding form for $E_{2m+1,s}$ is:

$$E_{2m+1,s} = \frac{\pi}{4(m+1)} \frac{1}{\Gamma(\frac{3}{2}+s)\Gamma(\frac{1}{2}-s)} + \frac{\pi}{4} \frac{m!}{\Gamma(\frac{1}{2}+s)\Gamma(-\frac{1}{2}-s)} \times \\ \sum_{k=0}^{\infty} \frac{(\frac{3}{2}+s)_k (\frac{1}{2}-s)_k}{k!(k+m+2)!} \left\{ \Psi\left(\frac{3}{2}+s+k\right) + \Psi\left(\frac{1}{2}-s+k\right) - \Psi(k+1) - \Psi(3+m+k) \right\}.$$

This, however, can be replaced by

$$E_{2m-1,s} = \frac{\cos \pi s}{2(s+m)+1} \left\{ \frac{1}{2s+1} + (2s+1) \sum_{k=0}^{m-1} \frac{(m-1)!^2}{(m-1-k)!(m+k)!} \frac{2k+1}{(2k+1)^2 - 4s^2} \right\}, \quad (32)$$

on combining (24) with (78) below. \diamond

Example 3 (Other special values) For each $s \neq 0$ there are also two special values of r for which $K_{r,s}$ also reduce to a ${}_2F_1$. They are obtained by solving $r + 3/2 = 1/2 \pm s$. This and similar calculations for $E_{n,s}$ yield

$$K_{(-2 \pm 2s),s} = -\frac{\cos \pi s}{(1 \mp 2s)^2}, \quad (33)$$

$$E_{(-2-2s),s} = -\frac{2}{(1+2s)} \frac{\cos(\pi s)}{(1-2s)^2}, \quad (34)$$

$$E_{(-4-2s),s} = -\frac{2}{(1+2s)} \frac{\cos(\pi s)}{(3+2s)^2}. \quad (35)$$

The r -recursions given above in (12) for $K_{r,s}$ and below in equation (78) for $E_{r,s}$ extend this to values of $r + 2n$, for n integral. \diamond

Example 4 (Alternative moment expansions) We also obtain

$$K_{0,s} = \frac{\cos(\pi s)}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+s)_n (\frac{1}{2}-s)_n}{n! (\frac{3}{2})_n} \times \\ \left\{ \Psi(n+1) + \Psi\left(\frac{3}{2}+n\right) - \Psi\left(\frac{1}{2}+n+s\right) - \Psi\left(\frac{1}{2}+n-s\right) \right\}, \\ E_{0,s} = \frac{\cos \pi s}{2s+1} + \cos \pi s \frac{2s+1}{6} \sum_{n=0}^{\infty} \frac{(\frac{3}{2}+s)_n (\frac{1}{2}-s)_n}{n! (\frac{5}{2})_n} \times \\ \left\{ \Psi(n+1) + \Psi\left(\frac{5}{2}+n\right) - \Psi\left(\frac{3}{2}+n+s\right) - \Psi\left(\frac{1}{2}+n-s\right) \right\}.$$

\diamond

2.1.1 Half-integer values of s

For $s = m + 1/2$, and $m, n = 0, 1, 2, \dots$ we can obtain a terminating representation

$$\begin{aligned} K_{n,m+1/2} &= \frac{\pi}{2(n+1)} {}_3F_2 \left(\begin{matrix} -m, m+1, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix} \middle| 1 \right) \\ &= \frac{(-1)^m \pi}{4} \frac{\Gamma^2 \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n+1}{2} - m \right) \Gamma \left(\frac{n+3}{2} + m \right)}, \end{aligned} \quad (36)$$

and likewise

$$E_{n,m+1/2} = \frac{\pi}{2} \sum_{k=0}^{m+1} \frac{(-m-1)_k (m+1)_k}{(n+1+2k) k!^2}. \quad (37)$$

2.2 The complementary integrals

By contrast, the complementary integral moments are somewhat less recondite.

Theorem 4 (Complementary moments) For $n = 0, 1, 2, \dots$ and $0 \leq s < \frac{1}{2}$ we have

$$K'_{n,s} = \frac{\pi}{4} \frac{\Gamma^2 \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n+2-2s}{2} \right) \Gamma \left(\frac{n+2+2s}{2} \right)} \quad (38)$$

$$E'_{n,s} = \frac{\pi}{2(n+1)} \frac{\Gamma^2 \left(\frac{n+3}{2} \right)}{\Gamma \left(\frac{n+2-2s}{2} \right) \Gamma \left(\frac{n+4+2s}{2} \right)}. \quad (39)$$

These hold in the limit for $s = \frac{1}{2}$.

In particular, recursively we obtain for all real n :

$$(a) \quad K'_{n+2,s} = \frac{(n+1)^2}{(n+2)^2 - 4s^2} K'_{n,s}, \quad (b) \quad E'_{n,s} = \frac{n+1}{n+2+2s} K'_{n,s}, \quad (40)$$

$$\text{where (c) } K'_{0,s} = \frac{\pi}{4} \frac{\sin(\pi s)}{s}, \quad (d) \quad K'_{1,s} = \frac{\cos \pi s}{1 - 4s^2}.$$

Proof. To establish (38) we recall that

$$K^{s'} = \frac{\pi}{2} {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| 1 - k^2 \right), \quad (41)$$

and so

$$\begin{aligned}
K'_{n,s} &= \frac{\pi}{2} \int_0^1 x^n {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| 1 - x^2 \right) dx \\
&= \frac{\pi}{4} \int_0^1 x^{\frac{n+1}{2}-1} {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| 1 - x \right) dx \\
&= \frac{\pi}{4} \int_0^1 (1-x)^{\frac{n+1}{2}-1} {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| x \right) dx \\
&= \frac{\pi}{2(n+1)} {}_3F_2 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s, 1 \\ 1, \frac{n+3}{2} \end{matrix} \middle| 1 \right) \\
&= \frac{\pi}{2(n+1)} {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ \frac{n+3}{2} \end{matrix} \middle| 1 \right),
\end{aligned}$$

which is summable, by Gauss' formula (3), to the desired result.

The proof of (39) is similar, and the recursions follow. \square

Example 5 (Complementary closed forms) Thence, with $s = 0$ and $n = 0, 1$ we recover

$$K'_0 = \frac{\pi^2}{4}, \quad E'_0 = \frac{\pi^2}{8}, \quad K'_1 = 1, \quad E'_1 = \frac{2}{3},$$

as discussed in [7, p. 198]. Correspondingly

$$\begin{aligned}
K'_{0,1/6} &= \frac{3\pi}{4}, & K'_{1,1/6} &= \frac{9\sqrt{3}}{16}, & E'_{0,1/6} &= \frac{9\pi}{28}, & K'_{1,1/6} &= \frac{27\sqrt{3}}{80}, \\
K'_{0,1/3} &= \frac{3\sqrt{3}\pi}{8}, & K'_{1,1/3} &= \frac{9}{10}, & E'_{0,1/3} &= \frac{9\sqrt{3}\pi}{64}, & E'_{1,1/3} &= \frac{27}{55}.
\end{aligned}$$

We note that π , not π^2 appears in these evaluations, since in (40, c), $\sin(\pi s)/s \rightarrow \pi$ as $s \rightarrow 0$. \diamond

2.2.1 Connecting moments and complementary moments

We first remark that a comparison of Theorems 3 and 4 shows that for all s we have

$$K'_{1,s} = K_{1,s} \quad \text{and} \quad E'_{1,s} = E_{1,s}.$$

The formula

$$\int_0^1 K(k) \frac{dk}{1+k} = \int_0^1 K\left(\frac{1-h}{1+h}\right) \frac{dh}{1+h} = \frac{1}{2} \int_0^1 K'(k) dk \quad (42)$$

is recorded in [7, p. 199]. It is proven by using the quadratic transform [7, Thm 1.2 (b), p. 12] for the second equality and a substitution for the first. This implies

$$2 \sum_{n=0}^{\infty} (-1)^n K_n = \frac{\pi^2}{4} = K'_0, \quad (43)$$

on appealing to Theorem 4.

The corresponding identity for $s = 1/6$ is best written

$$\int_0^1 {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| 1-t^3\right) dt = 3 \int_0^1 {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| t^3\right) \frac{dt}{1+2t}, \quad (44)$$

which follows analogously from the cubic transformation [9, Eqn 2.1] and a change of variables. This is a beautiful counterpart to (42) especially when the latter is written in hypergeometric form:

$$\int_0^1 {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| 1-k^2\right) dk = 2 \int_0^1 {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| k^2\right) \frac{dk}{1+k}. \quad (45)$$

We further evaluate equation (44) in (99) of section 5.4.

Additionally, [7, p. 188] outlines how to derive

$$\int_0^1 \frac{K(k) dk}{\sqrt{1-k^2}} = K\left(\frac{1}{\sqrt{2}}\right)^2.$$

Using the same technique, we generalize this to

$$\int_0^1 \frac{K^s(k) dk}{\sqrt{1-k^2}} = K^s\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\cos^2(\pi s)}{16\pi} \Gamma^2\left(\frac{1+2s}{4}\right) \Gamma^2\left(\frac{1-2s}{4}\right). \quad (46)$$

Here we have used Gauss' formula (3) for the evaluation

$$K^s\left(\frac{1}{\sqrt{2}}\right) = \frac{\cos \pi s}{4} \beta\left(\frac{1+2s}{4}, \frac{1-2s}{4}\right).$$

By the generalized Legendre's identity (13), which simplifies as the complementary integrals coincide with the original ones at $1/\sqrt{2}$, we obtain

$$E^s\left(\frac{1}{\sqrt{2}}\right) = \frac{K^s\left(\frac{1}{\sqrt{2}}\right)}{2} + \frac{\pi \cos \pi s}{4(2s+1)K^s\left(\frac{1}{\sqrt{2}}\right)}.$$

2.3 Analytic continuation of results

We finish this section by recalling a useful theorem:

Theorem 5 (Carlson (1914)) *Let f be analytic in the right half-plane $\Re z \geq 0$ and of exponential type (meaning that $|f(z)| \leq Me^{c|z|}$ for some M and c), with the additional requirement that*

$$|f(z)| \leq Me^{d|z|}$$

for some $d < \pi$ on the imaginary axis $\Re z = 0$. If $f(k) = 0$ for $k = 0, 1, 2, \dots$ then $f(z) = 0$ identically.

Carlson's dissertation result [15, 5.81] allows us to prove that many of the results in this paper hold generally as soon as they hold for integer n . For example, the equations (75) or (76) hold generally as soon as the integral cases hold: once we check growth on the imaginary axis which is easy for hypergeometric functions. This matter is discussed at some length in [3, Thm 2.8.1 and sequel] — including an elegant 1941 proof by Selberg for the case where f is bounded in the right half-plane.

3 Closed form initial-values for various s

Many results work for all s (as we have seen) but a few others are more satisfactory when $s \in \Omega$ — since these four K^s are the only modular functions ([7, Prop 5.7], [9]) amongst the generalized elliptic integrals K^s .

Empirically, we discovered an algebraic relation

$$2(1+s)E_{0,s} - (1+2s)K_{0,s} = \frac{\cos \pi s}{1+2s}. \quad (47)$$

Equivalently, we exhibit a parametric series for $1/\pi$:

$$\frac{1}{\pi} = \frac{(1+2s)(2+2s) {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}+s, -\frac{1}{2}-s \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right) - (1+2s)^2 {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}+s, \frac{1}{2}-s \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right)}{2 \cos(\pi s)}.$$

On using (11) to eliminate $E_{0,s}$ in (47), it becomes

$$K_{2,s} = \frac{K_{0,s} + \cos(\pi s)}{4 - 4s^2} \quad (48)$$

which in turn is a special case of (76) with $r = \frac{1}{2}$ (as is justified by Carlson's Theorem 5), thus proving our empirical observation.

Hence, to resolve all integral values for a given s , we are left with looking for satisfactory representations only for $K_{0,s}$. We will write

$$G_s := \frac{1}{2}K_{0,s} = \frac{\pi}{4} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} - s, \frac{1}{2} + s \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right).$$

and call this the associated or *generalized Catalan* constant. For various reasons, the results for $s = 1/6$ are especially interesting. This is the case corresponding to the cubic AGM [9].

3.1 Evaluation of G_s

From (20) we obtain

$$\begin{aligned} K_{0,s} &= \frac{\pi}{2} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} - s, \frac{1}{2} + s \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{\cos \pi s}{2} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + n + s) \Gamma(\frac{1}{2} + n - s)}{(2n+1)(n!)^2} \\ &= \frac{\cos \pi s}{2} \sum_{n=0}^{\infty} \beta \left(n + \frac{1}{2} - s, n + \frac{1}{2} + s \right) \frac{\binom{2n}{n}}{2n+1} \\ &= \frac{\cos \pi s}{4} \int_0^1 \frac{\arcsin(2\sqrt{t-t^2})}{t^{1+s}(1-t)^{1-s}} dt \\ &= \frac{\cos \pi s}{2} \int_0^{\pi/2} \left\{ \tan^{2s} \left(\frac{\theta}{2} \right) + \cot^{2s} \left(\frac{\theta}{2} \right) \right\} \frac{\theta}{\sin \theta} d\theta. \end{aligned} \quad (49)$$

This uses the definition directly, see also [7, Prop 5.6], to attain the first identity after writing the rising factorials in terms of the β function, whose integral representation we use here:

$$\beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt.$$

We exchange integral and sum to arrive at the penultimate integral. Moving the integral to $[-1/2, 1/2]$ and then making various trig substitutions, we arrive at the final result in (49). For example, we have

$$K_{0,0} = \int_0^{\pi/2} \frac{\theta}{\sin \theta} d\theta = 2G.$$

The final equality has various derivations [7, 1]. These include contour integration as explored in section 4.

If we now make the trigonometric substitution $t = \tan(\theta/2)$ in (49), and integrate the two resulting terms separately, we arrive at a central result.

Theorem 6 (Generalized Catalan constants for $0 \leq s \leq \frac{1}{2}$)

$$\begin{aligned} K_{0,s} &= \cos \pi s \int_0^1 (t^{2s-1} + t^{-2s-1}) \arctan t dt \\ &= \frac{\cos \pi s}{8s} \left\{ \Psi\left(\frac{3-2s}{4}\right) + \Psi\left(\frac{1+2s}{4}\right) - \Psi\left(\frac{1-2s}{4}\right) - \Psi\left(\frac{3+2s}{4}\right) \right\} \end{aligned} \quad (50)$$

$$= \frac{\cos \pi s}{4s} \left\{ \Psi\left(\frac{s}{2} + \frac{1}{4}\right) - \Psi\left(\frac{s}{2} + \frac{3}{4}\right) \right\} + \frac{\pi}{4s} = 2G_s. \quad (51)$$

Note that for $s = 0$, applying L'Hôpital's rule to (50) yields

$$K_{0,0} = \frac{1}{8}\Psi'\left(\frac{1}{4}\right) - \frac{1}{8}\Psi'\left(\frac{3}{4}\right)$$

which is precisely $2G$.

The digamma expression in (51) simplifies entirely when $s \in \Omega$ to the forms originally discovered in the next section. We now obtain complete evaluations for $s \in \Omega$, as was our goal.

Corollary 1 (Generalized Catalan values for s in Ω)

$$G_0 = G, \quad G_{1/6} = \frac{3}{4}\sqrt{3} \log 2, \quad G_{1/4} = \log\left(1 + \sqrt{2}\right), \quad G_{1/3} = \frac{3}{8}\sqrt{3} \log\left(2 + \sqrt{3}\right). \quad (52)$$

Mathematica, which currently knows more about the Ψ function than *Maple*, can evaluate the integral in Theorem 6 symbolically for some s . For example, if $s = 1/12$, after simplification we have the very nice expression:

$$G_{1/12} = 3\left(\sqrt{3} + 1\right) \left\{ \log\left(\sqrt{2} - 1\right) + \frac{\sqrt{3}}{2} \log\left(\sqrt{3} + \sqrt{2}\right) \right\}.$$

More generally, the evaluation requires only knowledge of $\sin(\pi s/2)$, and hence we can determine which s give a reduction to radicals. As a last example,

$$G_{1/5} = \frac{5}{8} \sqrt{5 + 2\sqrt{5}} \left\{ \frac{\sqrt{5} - 1}{2} \operatorname{arcsinh} \left(\sqrt{5 + 2\sqrt{5}} \right) - \operatorname{arcsinh} \left(\sqrt{5 - 2\sqrt{5}} \right) \right\}.$$

3.2 Other generalizations of G

Two other famous representations of G are:

$$G = - \int_0^{\pi/2} \log \left(2 \sin \frac{t}{2} \right) dt \quad (53)$$

$$= \int_0^{\pi/2} \log \left(2 \cos \frac{t}{2} \right) dt \quad (54)$$

and

$$G = - \int_0^{\pi/2} \log (\tan t) dt, \quad (55)$$

which easily follows from (53) and (54). To prove (53) we integrate by parts and obtain

$$\begin{aligned} - \int_0^{\pi/2} \log \left(2 \sin \frac{t}{2} \right) dt &= 2 \int_0^{\pi/4} t \cot t dt - \frac{\pi}{4} \log 2 \\ &= 2 \int_0^{\pi/4} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2} \middle| \sin^2 t \right) \cos t dt - \frac{\pi}{4} \log 2 \\ &= 2 \int_0^{1/\sqrt{2}} \frac{\arcsin x}{x} dx - \frac{\pi}{4} \log 2 \\ &= \left(G + \frac{\pi}{4} \log 2 \right) - \frac{\pi}{4} \log 2 = G. \end{aligned}$$

The second and third equalities hold since ${}_2F_1 \left(\frac{1}{2}, \frac{1}{2} \middle| x^2 \right) = \arcsin x$. The final equality follows on integrating $\arcsin(x)/x$ term by term. *Inter alia*, we have shown that

$$G = \int_0^{\pi/2} \frac{t}{\sin t} dt = \int_0^{\pi/2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2} \middle| \sin^2 t \right) dt. \quad (56)$$

We may generalize (53) or equivalently (56) to:

Proposition 1

$$G_s = \frac{\cos \pi s}{2} \int_0^{\pi/2} \tan^{2s} t {}_2F_1 \left(\frac{1}{2}, \frac{1}{2} - s \middle| \sin^2 t \right) dt. \quad (57)$$

Proof. We write

$$\begin{aligned}
G_s &= \frac{1}{2} \int_0^1 K^s(k) dk = \frac{\pi}{4} \int_0^1 {}_2F_1 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s \\ 1 \end{matrix} \middle| k^2 \right) dk \\
&= \frac{\cos \pi s}{4} \int_0^1 t^{s-1/2} (1-t)^{-s-1/2} dt \int_0^1 (1-k^2 t)^{s-1/2} dk \\
&= \frac{\cos \pi s}{4} \int_0^1 t^{s-1/2} (1-t)^{-s-1/2} {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} - s \\ \frac{3}{2} \end{matrix} \middle| t \right) dt \\
&= \frac{\cos \pi s}{2} \int_0^{\pi/2} \tan^{2s} u {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} - s \\ \frac{3}{2} \end{matrix} \middle| \sin^2 u \right) du.
\end{aligned}$$

□

Note that Theorem 2 gives a series for G_s for $0 \leq s \leq 1/2$:

$$\begin{aligned}
\frac{4}{\pi} G_s &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - s\right)_n \left(\frac{1}{2} + s\right)_n}{(n!)^2 (2n+1)} \\
&= {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} + s, \frac{1}{2} - s \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right).
\end{aligned} \tag{58}$$

Recalling (29) we recover Theorem 6 in the equivalent form

$$G_s = \frac{\pi}{4} {}_3F_2 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{\cos \pi s}{4s} \gamma \left(\frac{1}{2} + s \right) - \frac{\pi}{8s}. \tag{59}$$

From (58) it is clear that G_s is monotonically decreasing from G to $\pi/4$ as s runs from 0 to $1/2$. In fact, G_s is concave on $[0, 1/2]$, as illustrated in Figure 1.

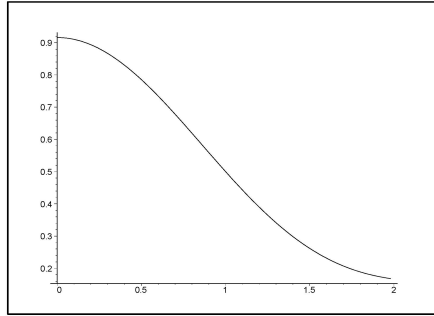


Figure 1: (58) plotted on $[0, 2]$.

4 Contour integrals for $K_{0,s}$

By contour integration on the infinite rectangle above $[0, \pi/2]$ we obtain

$$\begin{aligned} G_0 &= \frac{1}{2} \int_0^\infty \frac{t}{\cosh t} dt \\ &= \int_0^\infty \frac{te^{-t}}{1+e^{-2t}} dt = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} = G. \end{aligned} \quad (60)$$

Here we have used the geometric series and integrated term by term the Γ function terms that we obtain. The final evaluation is definitional.

Done carefully, contour integration over the same rectangle, converting to exponentials, and then integrating term by term, provides a fine general integral evaluation:

Theorem 7 (Contour integral for G_s) For $0 \leq s < 1/2$ we have

$$\begin{aligned} 2G_s = K_{0,s} &= 2^{2s} \sin(2\pi s) \int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}} t dt + \\ &\quad \cos(\pi s) \int_0^\infty \frac{\cos(2s \arctan(\sinh t))}{\cosh t} t dt. \end{aligned} \quad (61)$$

Example 6 (Experimentally obtained evaluations) For $s = 1/4$, equation (61) becomes

$$K_{0,1/4} = \sqrt{2} \int_0^\infty \frac{\cosh t - \sinh t}{(\sinh 2t)^{3/2}} t dt + 2\sqrt{2} \int_0^\infty \frac{\cosh t}{(\cosh 2t)^{3/2}} t dt, \quad (62)$$

with numerical value ≈ 1.7627471740392 . Here for the first time the specific form of the root of unity has played a role. Quite remarkably, if we — much as before — convert the integrand to exponential form and apply the binomial theorem, we obtain Γ function values which become:

$$\begin{aligned} G_{1/4} &= \sum_{n=0}^{\infty} \binom{-\frac{3}{2}}{n} \frac{12n + 8n^2 + 5 + (-1)^n (2n+1)^2}{8(n+1)^2 (2n+1)^2} \\ &= \log(1 + \sqrt{2}). \end{aligned} \quad (63)$$

Having first proven this, we then discovered using the integer relation algorithm PSLQ and the *Maple* identify function that:

$$K_{0,1/6} = \frac{3}{2} \sqrt{3} \log 2, \quad (64)$$

with numerical value ≈ 1.8008492007794 , and a similar evaluation:

$$K_{0,1/3} = \frac{3}{2} \sqrt{3} \log(1 + \sqrt{3}) - \frac{3}{4} \sqrt{3} \log(2), \quad (65)$$

with numerical value ≈ 1.7107784916770 . ◇

Example 7 (Further integrals) We have discovered additionally, using inverse symbolic computational methods (<http://carma.newcastle.edu.au/isc2>), that

$$\int_0^\infty \frac{(\cosh t)^{4/3} - (\sinh t)^{4/3}}{(\sinh t \cosh t)^{5/3}} t dt = \frac{9}{4} \log(3),$$

and

$$\int_0^\infty \frac{(\cosh t)^{2/3} - (\sinh t)^{2/3}}{(\sinh t \cosh t)^{4/3}} t dt = \frac{3}{2} \log\left(\frac{27}{16}\right).$$

In light of Corollary 1 these are now proven. \diamond

4.1 Contour integral based series for $K_{0,s}$

Let us write

$$K_{0,s} = \sin(2\pi s) S(s) + \cos(\pi s) C(s) \quad (66)$$

where

$$S(s) := 2^{2s} \int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}} t dt \quad (67)$$

$$C(s) := \int_0^\infty \frac{\cos(2s \arctan(\sinh t))}{\cosh t} t dt. \quad (68)$$

To evaluate $S(s)$ we make a substitution $u = \tanh(t)$. We obtain

$$\begin{aligned} S(s) &= \frac{1}{2} \int_0^1 (u^{-2s-1} - u^{2s-1}) \operatorname{arctanh}(u) du \\ &= \frac{-1}{8s} \left(2\gamma + 4 \log(2) + \Psi\left(\frac{1}{2} - s\right) + \Psi\left(\frac{1}{2} + s\right) \right). \end{aligned} \quad (69)$$

Here γ denotes the *Euler-Mascheroni* constant.

To evaluate $C(s)$ we note that

$$\cos(2s \arctan(\sinh t)) = \cos(2s \arcsin(\tanh t)) = {}_2F_1\left(\begin{matrix} s, -s \\ \frac{1}{2} \end{matrix} \middle| \tanh^2 t\right) \quad (70)$$

and so we obtain a converging (finite if $s = 0$) series

$$C(s) = \int_0^\infty \frac{\cos(2s \arctan(\sinh t))}{\cosh t} t dt = \sum_{n=0}^{\infty} \frac{(s)_n (-s)_n}{\left(\frac{1}{2}\right)_n} \frac{\tau_n}{n!}$$

where

$$\tau_n := \int_0^\infty \frac{x^{2n}}{(1+x^2)^{n+1}} \operatorname{arcsinh}(x) dx, \quad (71)$$

and where we have expanded termwise. Moreover,

$$\tau_{m+2} = \frac{(13 + 8m^2 + 20m)\tau_{m+1} - 2(m+1)(2m+1)\tau_m}{2(m+2)(2m+3)} \quad (72)$$

where $\tau_0 = K_0 = 2G$ and $\tau_1 = E_0 = G + \frac{1}{2}$. In particular $C(0) = 2G$.

A closed form for τ_n is easily obtained. It is

$$\tau_n = \beta\left(n + \frac{1}{2}, \frac{1}{2}\right) \left\{ \frac{2G}{\pi} + \frac{1}{4} \sum_{k=1}^n \frac{\Gamma(k)^2}{\Gamma(k + \frac{1}{2})^2} \right\}. \quad (73)$$

Collecting up evaluations, we deduce that

$$K_{0,s} = \sin(2\pi s) \left\{ \frac{-1}{8s} \left(2\gamma + 4\log(2) + \Psi\left(\frac{1}{2} - s\right) + \Psi\left(\frac{1}{2} + s\right) \right) \right\} + \frac{\sin(2\pi s)}{\pi s} \left\{ G - \pi \sum_{k=0}^{\infty} \frac{\Gamma(k+s+1)\Gamma(k-s+1) - k!^2}{8\Gamma(k + \frac{3}{2})^2} \right\},$$

since on interchanging order of summation

$$\frac{\pi}{4} \cos(\pi s) \sum_{n=1}^{\infty} \frac{(s)_n (-s)_n}{n!^2} \sum_{k=1}^n \frac{\Gamma(k)^2}{\Gamma(k + \frac{1}{2})^2} = -\frac{\sin 2\pi s}{8s} \sum_{k=1}^{\infty} \frac{\Gamma(k+s)\Gamma(k-s) - \Gamma(k)^2}{\Gamma(k + \frac{1}{2})^2}.$$

This ultimately yields:

Theorem 8 (Contour series for G_s)

$$G_s = \frac{\sin 2\pi s}{16s} \left(\sum_{k=1}^{\infty} \frac{\Gamma(k)^2 - \Gamma(k+s)\Gamma(k-s)}{\Gamma(k + \frac{1}{2})^2} + 2\Psi\left(\frac{1}{2}\right) - 2\Psi\left(s + \frac{1}{2}\right) + \pi \tan(\pi s) + \frac{8G}{\pi} \right). \quad (74)$$

Example 8 (A related series) Note for $s = 0$ we obtain precisely $G_0 = G$ as all other terms in (74) are zero. Comparing, (74) to (50) leads to a closed form for the infinite series $Q(s)$ given by

$$\begin{aligned} Q(s) &:= \sum_{k=1}^{\infty} \frac{\Gamma(k+s)\Gamma(k-s) - \Gamma(k)^2}{\Gamma(k + \frac{1}{2})^2} \\ &= \frac{8}{\pi} \int_0^{\pi/4} \frac{(\tan t)^{2s} + (\cot t)^{2s} - 2}{\cos 2t} t \, dt \\ &= \frac{8}{\pi} \int_0^1 \frac{(x^s - x^{-s})^2}{1 - x^2} \arctan x \, dx. \end{aligned}$$

The integrals above are obtained much as in the derivation of (74). For example,

$$Q\left(\frac{1}{4}\right) = \frac{8G}{\pi} - 4 \log\left(1 + \frac{1}{\sqrt{2}}\right),$$

and there other nice evaluations. ◇

5 Closed forms at negative integers

We observe that (20) and (21) give analytic continuations which allow us to study negative moments. In [1] Adamchik studies such moments of K .

5.1 Negative moments

Adamchik's starting point is the study of $K_n = K_{n,0}$ for which Ramanujan appears to have known that

$$(2r+1)^2 K_{2r+1} - (2r)^2 K_{2r-1} = 1, \quad (75)$$

for $\Re r > -1/2$. For integer r this is a direct consequence of (24).

Experimentally, we found the following extension for general s by using integer relation methods with $s := 1/n$ to determine the coefficients:

$$((2r+1)^2 - 4s^2) K_{2r+1,s} - (2r)^2 K_{2r-1,s} = \cos \pi s. \quad (76)$$

For integer r this is established as follows — the general case then follows by Carlson's Theorem 5. Using (24) and the functional relation for the Γ function, we have:

$$\begin{aligned} & ((2r+1)^2 - 4s^2) K_{2r+1,s} - 4r^2 K_{2r-1,s} \\ &= \frac{\pi (r!)^2}{\Gamma(\frac{1}{2} + r - s)\Gamma(\frac{1}{2} + r + s)} \left\{ \sum_{k=0}^r \frac{(\frac{1}{2} - s)_k (\frac{1}{2} + s)_k}{(k!)^2} - \sum_{k=0}^{r-1} \frac{(\frac{1}{2} - s)_k (\frac{1}{2} + s)_k}{(k!)^2} \right\} \\ &= \frac{\pi (r!)^2}{\Gamma(\frac{1}{2} + r - s)\Gamma(\frac{1}{2} + r + s)} \frac{(\frac{1}{2} - s)_r (\frac{1}{2} + s)_r}{(r!)^2} \\ &= \frac{\pi}{\Gamma(\frac{1}{2} - s)\Gamma(\frac{1}{2} + s)} = \cos(\pi s). \end{aligned}$$

From (76) by creative telescoping one again deduces

$$K_{2n+1,s} = \frac{\cos \pi s}{4} \frac{n!^2}{\Gamma(n + \frac{3}{2} + s)\Gamma(n + \frac{3}{2} - s)} \sum_{k=0}^n \frac{\Gamma(k + \frac{1}{2} + s)\Gamma(k + \frac{1}{2} - s)}{k!^2}. \quad (77)$$

This provides another proof of Theorem 3.

Equation (12), when combined with (76), implies

$$E_{n,s} = \frac{(2s+1)^2 K_{n,s} + \cos \pi s}{(2s+1)(2s+n+2)}, \quad (78)$$

which extends (16) and completes the proof in Example 2.

Adamchik also develops a reflection formula which in our terms is

$$K_{-1-2r}^* + K_{2r} = -\frac{\pi}{4^{2r}} \binom{2r}{r}^2 \{\log 2 + H_r - H_{2r}\} \quad (79)$$

for $r = 0, 1, 2, \dots$. Here

$$K_{-1-2r}^* := \lim_{t \rightarrow r} \left\{ K_{-1-2t} - \frac{\binom{2n}{n}^2}{4^{2n+1}} \frac{\pi}{t-r} \right\}. \quad (80)$$

Note that, as examined in Theorem 9 of the next subsection, K_{-2r-1}^* removes the singularity at $-2r-1$. Hence, it can be written as an infinite sum [1].

Example 9 (Terminating sums) While studying [1] we found the following results.

1. For $0 < a \leq 1$

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, a \\ 1, 1+a \end{matrix} \middle| 1 \right) = \frac{4a}{\pi} {}_3F_2 \left(\begin{matrix} 1, 1, 1-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right). \quad (81)$$

In particular when $a = 1/2$ then

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{2}{\pi} {}_3F_2 \left(\begin{matrix} 1, 1, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{4}{\pi} G, \quad (82)$$

$${}_3F_2 \left(\begin{matrix} \frac{3}{4}, 1, 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{\Gamma^4(1/4)}{16\pi}. \quad (83)$$

2. Moreover, for $n = 1, 2, 3, \dots$

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, n \\ 1, 1+n \end{matrix} \middle| 1 \right) \quad (84)$$

always terminates. For example,

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1 \\ 1, 2 \end{matrix} \middle| 1 \right) = \frac{4}{\pi}, \quad {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, 2 \\ 1, 3 \end{matrix} \middle| 1 \right) = \frac{40}{9\pi}. \quad (85)$$

3. Also for $n = 1, 2, \dots$

$$(2n+1)^2 {}_3F_2 \left(\begin{matrix} 1, 1, -n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) - 4n^2 {}_3F_2 \left(\begin{matrix} 1, 1, 1-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = 1, \quad (86)$$

$${}_3F_2 \left(\begin{matrix} 1, 1, 1-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{4^{2n-1}}{n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{4^{2k}}, \quad (87)$$

and

$${}_3F_2 \left(\begin{matrix} 1, 1, \frac{1}{2}-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{\binom{2n}{n}^2}{4^{2n}} \left\{ 2G + \sum_{k=0}^{n-1} \frac{4^{2k}}{\binom{2k}{k}^2 (2k+1)^2} \right\}. \quad (88)$$

4. For $0 < a \leq 1$ and $n = 1, 2, \dots$

$${}_3F_2 \left(\begin{matrix} 1, 1, 1-n-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{(a)_n^2}{(a + \frac{1}{2})_n^2} \left\{ {}_3F_2 \left(\begin{matrix} 1, 1, 1-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) + \frac{1}{4a^2} \sum_{k=0}^{n-1} \frac{(a + \frac{1}{2})_k^2}{(a+1)_k^2} \right\}, \quad (89)$$

and

$${}_3F_2 \left(\begin{matrix} 1, 1, -a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \left(\frac{2a}{2a+1} \right)^2 {}_3F_2 \left(\begin{matrix} 1, 1, 1-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) + \frac{1}{(2a+1)^2}. \quad (90)$$

5. Finally

$$\sum_{k=0}^n (-1)^k \frac{k!}{\Gamma^2(k + \frac{3}{2})(n-k)!} = \frac{n!}{\pi \Gamma^2(n + \frac{3}{2})} \sum_{k=0}^n \frac{\Gamma^2(k + \frac{1}{2})}{(k!)^2}. \quad (91)$$

◇

5.2 Analyticity of $K_{\cdot, s}$ for $0 \leq s < 1/2$

The analytic structure of $r \mapsto K_{r, s}$ is similar qualitatively for all values of s . This is illustrated in Figure 2 for $s = 1/3$ and $s = 1/\pi$ both superimposed on $s = 0$ (red). In all cases there are simple poles at odd negative integers with computable residues.

Theorem 9 (Poles of $K_{\cdot, s}$) *Let $R_{n, s}$ denote the residue of $K_{\cdot, s}$ at $r = -2n + 1$. Then*

$$(a) R_{n+1, s} = \frac{(n - \frac{1}{2})^2 - s^2}{n^2} R_{n, s}, \quad (b) R_{1, s} = \frac{\pi}{2}. \quad (92)$$

Explicitly

$$(c) R_{n, s} = \frac{\cos \pi s \Gamma(n - \frac{1}{2} + s) \Gamma(n - \frac{1}{2} - s)}{2 \Gamma^2(n)}. \quad (93)$$

Proof. Recursion (92, a) follows from multiplying (76) by $2(r+n) = (2r+1) - (1-2n) = (2r-1) - (-2n-1)$ and computing the limits as $r \rightarrow -n$.

Directly from Theorem 2, we have the

$$R_{1, s} = \frac{\pi}{2} \lim_{r \rightarrow -1} \frac{r+1}{r+1} {}_3F_2 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s, \frac{r+1}{2} \\ 1, \frac{r+3}{2} \end{matrix} \middle| 1 \right) = \frac{\pi}{2},$$

which is (b); part (c) follows easily as a telescoping product. □

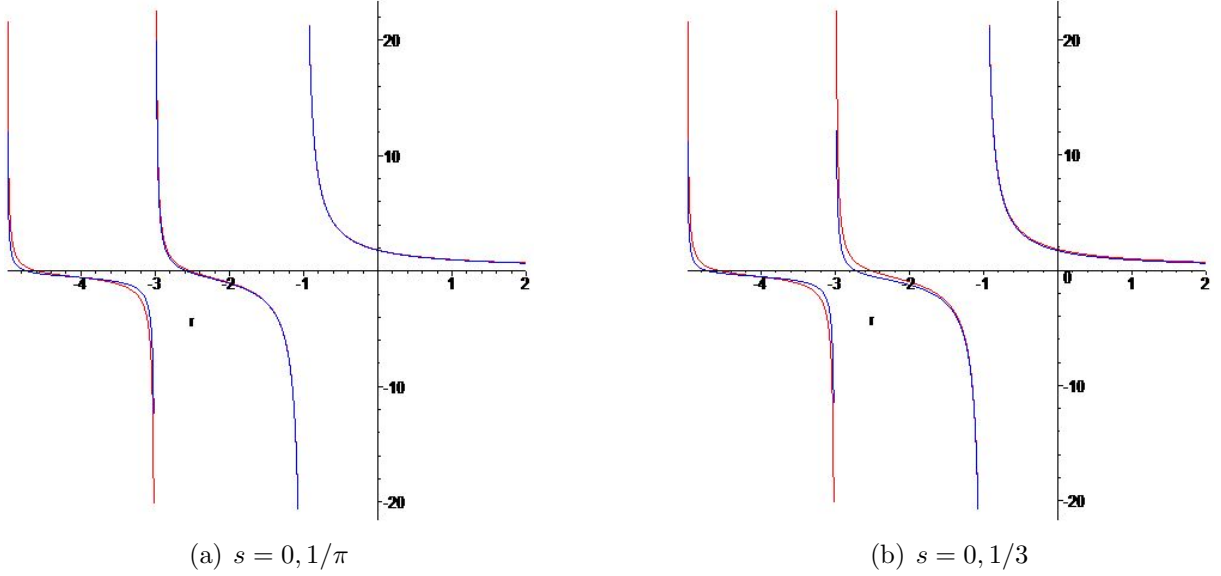


Figure 2: $r \mapsto K_{r,s}$ analytically continued to the real line.

5.3 Other rational values of s

Generally, directly integrating (1) or appealing to Theorem 2 yields the Saalschützian evaluation:

$$K_{(-1/2),s} = \pi {}_3F_2 \left(\begin{matrix} \frac{1}{2} + s, \frac{1}{2} - s, \frac{1}{4} \\ 1, \frac{5}{4} \end{matrix} \middle| 1 \right). \quad (94)$$

For $s = 0$ only, $K_{-1/2,s}$ reduces to a case of Dixon's theorem [14, Eqn. (2.3.3.5)] and yields

$$K_{(-1/2),0} = \frac{\Gamma(\frac{1}{4})^4}{16\pi}, \quad (95)$$

a result known to Ramanujan. Indeed, the two relevant specializations of Dixon's theorem are

$${}_3F_2 \left(\begin{matrix} \frac{1}{2} + s, \frac{1}{2} - s, \frac{1}{4} \\ 1 - 2s, \frac{5}{4} s - 1 \end{matrix} \middle| 1 \right) = \frac{\Gamma(\frac{5}{4} - \frac{1}{2}s) \Gamma(\frac{1}{2} - \frac{3}{2}s) \Gamma(1 - 2s) \Gamma(\frac{5}{4} - s)}{\Gamma(\frac{3}{2} - s) \Gamma(\frac{3}{4} - 2s) \Gamma(\frac{3}{4} - \frac{3}{2}s) \Gamma(1 - \frac{1}{2}s)}$$

and more pleasingly,

$${}_3F_2 \left(\begin{matrix} \frac{1}{4}, \frac{1}{2} - s, \frac{1}{2} + s \\ \frac{3}{4} + s, \frac{3}{4} - s \end{matrix} \middle| 1 \right) = \frac{\sqrt{2}\pi}{\Gamma^2(\frac{5}{8})} \frac{\Gamma(\frac{3}{4} + s) \Gamma(\frac{3}{4} - s)}{\Gamma(\frac{5}{8} + s) \Gamma(\frac{5}{8} - s)}.$$

In the same way, we should like to be able to evaluate $K_{-1/3,1/6}$ and $K'_{-1/3,1/6}$ or equivalently

$$H_0 = \frac{\pi}{2} \int_0^1 {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| t^3 \right) dt \quad \text{and} \quad H_0^* = \frac{\pi}{2} \int_0^1 {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| 1 - t^3 \right) dt, \quad (96)$$

respectively. So far we have met with partial success, see (97) and (99) below.

5.4 Moments with respect to t^3 instead

To evaluate H_0^* we first write

$$H_0^* = \frac{\pi}{6} \int_0^1 x^{-\frac{2}{3}} {}_2F_1 \left(\begin{matrix} \frac{\pi}{6}, \frac{2}{3} \\ 1 \end{matrix} \middle| 1-x \right) dx = \frac{\pi}{6} \int_0^1 (1-x)^{-\frac{2}{3}} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| x \right) dx.$$

Now the integral (22) shows this is $\frac{\pi}{2} {}_3F_2 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3}, 1 \\ \frac{2}{3}, \frac{4}{3} \end{matrix} \middle| 1 \right) = \frac{\pi}{2} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, 1 \\ \frac{4}{3} \end{matrix} \middle| 1 \right)$. By Gauss' formula (3) we arrive at

$$H_0^* = \frac{\pi}{2} \frac{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} = \frac{\sqrt{3}}{12} \Gamma^3 \left(\frac{1}{3} \right). \quad (97)$$

This also follows directly from the analytic continuation of the formula in (38) of Theorem 4. Similarly,

$$H_0 = \frac{\pi}{6} \int_0^1 x^{-\frac{2}{3}} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| x \right) dx = \frac{\pi}{3} {}_3F_2 \left(\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \\ 1, \frac{4}{3} \end{matrix} \middle| 1 \right).$$

If we use Bailey's identity:

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| 1 \right) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)} \frac{\Gamma(s)}{\Gamma(b+s)\Gamma(c+s)} {}_3F_2 \left(\begin{matrix} d-a, e-a, s \\ s+b, s+c \end{matrix} \middle| 1 \right)$$

for $s = d + e - a - b - c$, when $\text{Re}(s > 0)$, $\text{Re}(a) > 0$ [14, Eqn. (2.3.3.7)], this can be transformed to

$$H_0 = \frac{\pi}{6} \frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{16} {}_3F_2 \left(\begin{matrix} 1, 1, 1 \\ \frac{5}{3}, \frac{5}{3} \end{matrix} \middle| 1 \right)$$

which seems more promising. Next, applying (16.4.11) in the *Digital Library of Math Functions*

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| 1 \right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} {}_3F_2 \left(\begin{matrix} a, d-b, d-c \\ d, d+e-b-c \end{matrix} \middle| 1 \right),$$

we arrive at

$$H_0 = \frac{\pi}{6} \frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{4} \sum_{n=1}^{\infty} \frac{\prod_{k=1}^n \frac{3k-1}{3k+1}}{3n+2}, \quad (98)$$

while

$$G = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^n \frac{1-2k}{1+2k}}{2n+1}.$$

Finally, we also arrive at a reworking of equation (44):

$$3 \sum_{k=0}^{\infty} (-2)^k H_k = 3 H_0^* = \frac{\sqrt{3}}{4} \Gamma^3 \left(\frac{1}{3} \right), \quad (99)$$

as a companion to (43).

6 Conclusion and open questions

Another impetus for this study was a query from Roberto Tauraso regarding whether, for integer $m = 0, 1, 2, \dots$, one can find closed forms for

$$T(m, s) := \sum_{k=1}^{\infty} \frac{(\frac{1}{2} + s)_k (\frac{1}{2} - s)_k}{(1)_k^2} \frac{1}{k^m}. \quad (100)$$

We are able to write, more generally, that

$$T(m, s, \alpha) := \sum_{k=1}^{\infty} \frac{(\frac{1}{2} + s)_k (\frac{1}{2} - s)_k}{(1)_k^2} \frac{1}{(k + \alpha)^m} \quad (101)$$

$$= \frac{\frac{1}{4} - s^2}{(\alpha + 1)^m} {}_{m+2}F_{m+1} \left(\begin{matrix} \frac{3}{2} + s, \frac{3}{2} - s, \alpha + 1, \dots, \alpha + 1 \\ 2, \alpha + 2, \dots, \alpha + 2 \end{matrix} \middle| 1 \right). \quad (102)$$

- Sad to say, we have nothing better to provide than the hypergeometric form of (102).
- We should also very much like to know if one can evaluate the cubic moment $H_0 = \frac{2}{3} K_{-1/3, 1/6}$ other than in (96), (98) as we were able to do for $K_{-1/2, 0}$. Both reduce to evaluation of cases of $\frac{\pi}{1+2s} {}_3F_2 \left(\begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s, \frac{s}{2} + \frac{1}{4} \\ 1, \frac{s}{2} + \frac{5}{4} \end{matrix} \middle| 1 \right)$ ($s = 0, 1/6$).
- Are there other non-trivial explicit fractional evaluations?
- What is the correct s -generalization of the reflection formula (80)?
- Finally, how do the connection results of (43), (99) generalize?

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