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Uniscalar p-adic Lie groups

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Abstract. A totally disconnected, locally compact group G is said to be *uniscalar* if its scale function $s_G : G \to \mathbb{N}$, as defined in [G. A. Willis, *The structure of totally disconnected, locally compact groups*, Math. Ann. **300** (1994), 341–363], is identically 1. It is known that G is uniscalar if and only if every element of G normalizes some open, compact subgroup of G. We show that every identity neighbourhood of a compactly generated, uniscalar p-adic Lie group contains an open, compact, normal subgroup. In contrast, uniscalar p-adic Lie groups which are not compactly generated need not possess open, compact, normal subgroups.

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1 Introduction

Following Palmer [8], we say that a totally disconnected, locally compact group G is *uniscalar* if its scale function $s_G : G \to \mathbb{N}$ is identically 1, or, equivalently, if every element $x \in G$ normalizes some open, compact subgroup U of G (depending on x). This article is devoted to the study of uniscalar p-adic Lie groups. We are interested in the question whether the existence of the open, compact subgroups U normalized by individual group elements forces the existence of an open, compact subgroup normalized by all group elements simultaneously, *i.e.*, the existence of an open, compact, normal subgroup. A counterexample shows that this need not be so if the uniscalar p-adic Lie group is not compactly generated (Section 6). For compactly generated groups however, the above question has a positive answer. Calling a topological group *pro-discrete* if its filter of identity neighbourhoods has a basis of open, compact, normal subgroups, we can even prove the following stronger assertion (Theorem 5.2):

$(*)$ *Every compactly generated, uniscalar p-adic Lie group is pro-discrete.*

We begin our studies with a characterization of uniscalar p-adic Lie groups: a p-adic Lie group G is uniscalar if and only if $\mathrm{Ad}(G)$ is a periodic subgroup of $\mathrm{Aut}(\mathrm{L}(G))$

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(Corollary 3.2). Here, a topological group H is said to be *periodic* if every $x \in H$ is a periodic element, an element $x \in H$ being called *periodic* if for every identity neighbourhood U in H, there exists some $n \in \mathbb{N}$ such that $x^n \in U$. (If H is locally compact, an element $x \in H$ is periodic if and only if the closed subgroup it generates is compact). Next, we show that a compactly generated p-adic Lie group G is pro-discrete if and only if $\mathrm{Ad}(G)$ is a relatively compact subgroup of $\mathrm{Aut}(\mathrm{L}(G))$ (Proposition 4.1).

After these reduction steps, Assertion $(*)$ follows from the fact that every compactly generated, periodic subgroup of $\mathrm{GL}(\mathrm{L}(G))$ is relatively compact (Parreau [9]).[2]

In a final section, we investigate consequences of our results for the structure of compactly generated locally compact, totally disconnected groups beyond the p-adic setting.

2 Prerequisites and notational conventions

We make essential use of the theory of scale functions on totally disconnected, locally compact groups, as developed in [11]–[14] (see also [7]). If G is a totally disconnected, locally compact group, its scale function $s_G : G \to \mathbb{N}$ is defined via

$$s_G(x) := \min\{[U : U \cap x^{-1}Ux] : U \text{ is a compact, open subgroup of } G\}$$

for $x \in G$.

A compact, open subgroup U of G is called *tidy for* x if $s_G(x) = [U : U \cap x^{-1}Ux]$, *i.e.*, if the minimum is attained at U.[3] Occasionally, we shall simply write s for s_G if no confusion is possible. An element $x \in G$ normalizes an open, compact subgroup of G if and only if $s_G(x) = s_G(x^{-1}) = 1$ (cf. [12], Section 2). Hence G has the property that every $x \in G$ normalizes some open, compact subgroup if and only if $s_G \equiv 1$, *i.e.*, if and only if G is *uniscalar*. The scale functions of p-adic Lie groups can be computed in terms of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathrm{L}(G))$ ([3], Corollary 3.6):

Theorem 2.1. *Let G be a p-adic Lie group, and $x \in G$. Given a splitting field K for the characteristic polynomial of $\mathrm{Ad}(x)$, let $|.| : K \to \mathbb{R}_0^+$ be the unique extension of the absolute value $|.|_p := \mathrm{mod} : \mathbb{Q}_p \to \langle p \rangle \cup \{0\}$ on $\mathbb{Q}_p$ to an absolute value on K. Let $\lambda_1, \ldots, \lambda_n \in K$ be the eigenvalues of $\mathrm{Ad}(x) \otimes \mathrm{id}_K$, occurring with their proper multiplicities, and $I := \{i \in \{1, \ldots, n\} : |\lambda_i| \geq 1\}$. Then $s_G(x) = \prod_{i \in I} |\lambda_i|$. Furthermore, $\mathrm{im}\, s_G \subseteq p^{\mathbb{N}_0}$.* □

See [2], [10] for the prerequisites concerning p-adic Lie theory.

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[2] It is also recorded in [9] that our considerations and Parreau's combine to the proof of $(*)$ (*loc. cit.*, Corollaire 3).

[3] See [13], Theorem 3.1 for the equivalence of these definitions with the more complicated definitions given in the earlier paper [11].

3 Characterization of uniscalar groups

In this section, we show that a p-adic Lie group G is uniscalar if and only if $\mathrm{Ad}(G)$ is a periodic subgroup of $\mathrm{Aut}(\mathrm{L}(G))$.

First, let us recall the concept of a Campbell-Hausdorff group. Suppose that $\mathfrak{g}$ is a finite-dimensional $\mathbb{Q}_p$-Lie algebra. Let U be an open zero-neighbourhood in $\mathfrak{g}$ which is of the form $U = \{x \in \mathfrak{g} : \|x\| < r\}$ for some $r \in \,]0, p^{1/(p-1)}[$ and some norm $\|.\|$ on $\mathfrak{g}$ making $\mathfrak{g}$ a normed Lie algebra (*i.e.*, a norm such that $\|[a,b]\| \leq \|a\| \cdot \|b\|$ for all $a, b \in \mathfrak{g}$). Then the Campbell-Hausdorff series converges absolutely on $U \times U$ and defines a continuous multiplication $* : U \times U \to U$ making $(U, *)$ a p-adic Lie group, with 0 as the identity element ([2], §4.2, Lemma 3(iii)). Groups of this form are called *Campbell-Hausdorff groups*.

Proposition 3.1. *Let G be a p-adic Lie group, s its scale function, and $x \in G$. Then the following conditions are equivalent:*

1) $s(x) = s(x^{-1}) = 1$*;*

2) *x normalizes some open, compact subgroup of G;*

3) *there are small open, compact subgroups normalized by x (i.e., every identity neighbourhood of G contains an open, compact subgroup normalized by x).*

4) $\mathrm{Ad}(x) \in \mathrm{Aut}(\mathrm{L}(G))$ *is a periodic element.*

Proof. '1) $\Rightarrow$ 4)': Let K be a splitting field for the characteristic polynomial of $\mathrm{Ad}(x)$. By Theorem 2.1, the hypothesis $s(x) = s(x^{-1}) = 1$ implies that all eigenvalues of $\mathrm{Ad}(x)$ in K have modulus 1. Hence the semisimple part of $\mathrm{Ad}(x)$ in its multiplicative Jordan decomposition is a compact element, and hence a periodic element, of $\mathrm{GL}(\mathrm{L}(G))$. Now every unipotent element of a general linear group is a periodic element in the p-adic setting, see [3], Lemma 4.1. We conclude that $\mathrm{Ad}(x) = \mathrm{Ad}(x)_s \mathrm{Ad}(x)_u$ is a periodic element as well.

'4) $\Rightarrow$ 3)': Let $\exp : U \to G$ be an exponential function, defined on an open neighbourhood U of 0 in $\mathrm{L}(G)$. Then U contains an open neighbourhood V of 0 such that $\mathrm{Ad}(x)(V) \subseteq U$ and

$$(1) \qquad \exp \circ \mathrm{Ad}(x)|_V^U = I_x \circ \exp|_V$$

holds, where $I_x : G \to G$ denotes the inner automorphism $y \mapsto xyx^{-1}$. Now suppose that W is an arbitrary identity neighbourhood in G. There exists a Campbell-Hausdorff group C in $\mathrm{L}(G)$, contained in $V \cap \exp^{-1}(W)$, such that $\exp(C)$ is an open, compact subgroup of G and $\exp|_C^{\exp(C)}$ is an isomorphism of topological groups from $(C, *)$ onto $\exp(C)$. Since $\langle \mathrm{Ad}(x) \rangle$ is relatively compact, [10] Part II, Chapter IV, Appendix 1 shows that there exists a lattice M in $\mathrm{L}(G)$ which is invariant under $\langle \mathrm{Ad}(x) \rangle$. There exists $n \in \mathbb{N}$ such that $p^n M \subseteq C$. Then $C' := \langle p^n M \rangle$ is an open, compact, $\langle \mathrm{Ad}(x) \rangle$-invariant subgroup of the Campbell-Hausdorff group C; by

Equation (1), the open, compact subgroup $\exp(C')$ of G is normalized by x (and it is contained in W, by construction).

The implication '3) $\Rightarrow$ 2)' is obvious, and '2) $\Rightarrow$ 1)' (and indeed '2) $\Leftrightarrow$ 1)') holds for every locally compact, totally disconnected group, as mentioned above. □

Corollary 3.2. *Let G be a p-adic Lie group. Then G is uniscalar if and only if* $\mathrm{Ad}(G)$ *is a periodic subgroup of* $\mathrm{Aut}(\mathrm{L}(G))$. □

The implication '1) $\Rightarrow$ 3)' of Proposition 3.1 means that if G is a p-adic Lie group and $x \in G$ an element such that $s(x) = s(x^{-1}) = 1$, then there are small tidy subgroups for this element x. This need not be the case for arbitrary locally compact, totally disconnected groups: for example, $\mathbb{Z}_p^{\mathbb{Z}} \times \{0\}$ is the only subgroup of $\mathbb{Z}_p^{\mathbb{Z}} \rtimes \mathbb{Z}$ (with the shift action) which is tidy for $(0, 1)$. However, the above property of p-adic Lie groups generalizes to *pro-p-adic Lie groups*, *i.e.*, locally compact groups G with the property that every identity neighbourhood U contains a closed normal subgroup N of G such that G/N is a p-adic Lie group (cf. [4]):

Corollary 3.3. *Let G be a pro-p-adic Lie group, s its scale function, and $x \in G$. Consider the conditions* 1), 2), *and* 3) *given in Proposition 3.1. Then* 1), 2), *and* 3) *are equivalent.*

Proof. Once we have proved '2) $\Rightarrow$ 3), all other implications are trivial. Let W be an arbitrary open identity neighbourhood in G, and N an open, compact subgroup of G which is normalized by x. Since G is pro-p-adic, there is a compact, normal subgroup K of G, contained in W, such that G/K is a p-adic Lie group. Let $q : G \to G/K$ denote the canonical quotient morphism. By compactness, there is an open identity neighbourhood V in G such that $VK \subseteq W$. Now $V' := q(V)$ is an open identity neighbourhood in G/K, and $q(N)$ is an open, compact subgroup of G/K which is normalized by $q(x)$. By Proposition 3.1, there exists an open, compact subgroup C' of G/K which is normalized by $q(x)$ and contained in V'. We set $C := q^{-1}(C')$; then C is an open, compact subgroup of G which is contained in $VK \subseteq W$ and normalized by x. □

4 Characterization of pro-discrete groups

In this section, we characterize those compactly generated p-adic Lie groups which are pro-discrete.

Proposition 4.1. *Let G be a compactly generated p-adic Lie group. Then the following conditions are equivalent:*

(a) *G is pro-discrete;*

(b) $\mathrm{Ad}(G)$ *is a relatively compact subgroup of* $\mathrm{Aut}(\mathrm{L}(G))$.

Proof. Let $\exp : U \to G$ be an injective exponential function, defined on some open, compact 0-neighbourhood U in $\mathrm{L}(G)$, and let K be a compact symmetric generating set for G. For every $x \in K$, there exists an open neighbourhood W_x of x in G and an open 0-neighbourhood $V_x \subseteq U$ in $\mathrm{L}(G)$ such that $\mathrm{Ad}(y)(V_x) \subseteq U$ for all $y \in W_x$ and $I_y \circ \exp|_{V_x} = \exp \circ \mathrm{Ad}(y)|_{V_x}^{U}$. By compactness, there exists a finite subset F of K such that $K \subseteq \bigcup_{x \in F} W_x$. Set $V := \bigcap_{x \in F} V_x$; then

$$(2) \qquad I_y \circ \exp|_V = \exp \circ \mathrm{Ad}(y)|_V^U$$

holds, for every $y \in K$.

Now suppose that G is pro-discrete. Then there exists an open, compact, normal subgroup H of G such that $H \subseteq \exp(V)$. Set $C := \exp^{-1}(H)$; this is an open, compact subset of $\mathrm{L}(G)$. Note that C is invariant under $\mathrm{Ad}(K)$, by Equation (2). Since Ad is a homomorphism and K generates G, we conclude that C is invariant under $\mathrm{Ad}(G)$. Let M denote the $\mathbb{Z}_p$-submodule of $\mathrm{L}(G)$ generated by C; then M is a lattice in $\mathrm{L}(G)$ which is invariant under $\mathrm{Ad}(G)$. Hence by [10] Part II, Chapter 4, Appendix 1, the subgroup $\mathrm{Ad}(G)$ of $\mathrm{GL}(\mathrm{L}(G))$ is relatively compact, as required.

If, conversely, $\mathrm{Ad}(G)$ is a relatively compact subgroup of $\mathrm{GL}(\mathrm{L}(G))$, there exists a lattice M in $\mathrm{L}(G)$ invariant under $\mathrm{Ad}(G)$. We claim that G is pro-discrete. To see this, let N be an arbitrary identity neighbourhood in G; we have to find an open, compact, normal subgroup H of G such that $H \subseteq N$. There exists an open, compact Campbell-Hausdorff group C in $\mathrm{L}(G)$ such that $C \subseteq \exp^{-1}(N) \cap V$ and such that $\exp|_C$ is an isomorphism onto an open, compact subgroup of G. We may assume that $M \subseteq C$ (otherwise we shrink M by multiplication with powers of p). Let C' denote the subgroup of C generated by M; this is an open, compact subgroup of C which is invariant under $\mathrm{Ad}(G)$. Then $H := \exp(C')$ is an open, compact subgroup of G, contained in N, and by Equation (2), the normalizer of H in G contains K, and hence is all of G.

□

5 The main theorem

In this section, we show that every compactly generated, uniscalar p-adic Lie group is pro-discrete.

We make essential use of a recent result by A. Parreau:

Proposition 5.1. *For every $n \in \mathbb{N}$, every compactly generated, periodic subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ is relatively compact.*

Proof. The proposition is a special case of [9], Théorème 1. For finitely generated subgroups, the result is also stated in *loc. cit.*, Introduction, in a formulation more closely adapted to our needs.[4] □

[4] We remark that Proposition 5.1 can be reduced to the finitely generated case ([5], Lemmas A1 and A8).

It only remains to combine our findings from Sections 3 and 4 with Parreau's result.

Theorem 5.2. *Compactly generated, uniscalar p-adic Lie groups are pro-discrete.*

Proof. Let G be a compactly generated, uniscalar p-adic Lie group. By Corollary 3.2, $\mathrm{Ad}(G)$ is a periodic subgroup of $\mathrm{GL}(\mathrm{L}(G))$, which is compactly generated since G is so. Proposition 5.1 entails that $\mathrm{Ad}(G)$ is a relatively compact subgroup of $\mathrm{GL}(\mathrm{L}(G))$. By Proposition 4.1, G is pro-discrete. □

Remark 5.3. In an earlier version of this paper dating back to 1997, the authors had already shown that Proposition 5.1 and Theorem 5.2 are equivalent, where Proposition 5.1 holds for a fixed n provided that every compactly generated periodic subgroup H of $\mathrm{PSL}_n(L_n)$ is relatively compact for a certain finite extension field L_n of $\mathbb{Q}_p$ (cf. [5], Appendix). Relative compactness of H is equivalent to the existence of a fixed point under the action of H on the Bruhat-Tits building associated with $\mathrm{PSL}_n(L_n)$. For $n = 2$, the existence of fixed points is guaranteed by a classical result by J.-P. Serre, so that the 2-dimensional case of Proposition 5.1 follows. Parreau's proof is based on an existence proof for fixed points in the Bruhat-Tits building associated with $\mathrm{GL}_n(\mathbb{Q}_p)$.

Remark 5.4. It is natural to ask whether Theorem 5.2 admits generalizations beyond the p-adic setting. In Section 7, we prove an analogue for pro-p-adic Lie groups. Presumably, one cannot get much further, as there is an example of a compactly generated, totally disconnected, locally compact group which is uniscalar but does not have a compact, open, normal subgroup ([1], [6]).

We conclude this section with an immediate consequence of Theorem 5.2:

Corollary 5.5. *If a compactly generated p-adic Lie group G has a compact, open, normal subgroup, then for every identity neighbourhood U of G, there exists a compact, open, normal subgroup N of G such that* $N \subseteq U$. □

6 Example of a uniscalar *p*-adic Lie group without open, compact, normal subgroups

Theorem 5.2 would become false if we dropped the hypothesis that the uniscalar p-adic Lie groups considered are compactly generated, since there are uniscalar p-adic Lie groups (necessarily not compactly generated) which do not possess open, compact, normal subgroups. We shall presently give a 1-dimensional example of such a group. Its construction uses the following lemma:

Lemma 6.1. *Let G be a one-dimensional p-adic Lie group, s its scale function, Δ its modular function, and* $x \in G$. *Then* $s(x) = 1$ *or* $s(x^{-1}) = 1$. *In particular,* $\ker \Delta = \{x \in G : s(x) = s(x^{-1}) = 1\}$.

Proof. By Theorem 2.1, we have

$$s(x) = \begin{cases} |\lambda|_p & \text{if } |\lambda|_p \geq 1 \\ 1 & \text{else,} \end{cases}$$

where $\mathrm{Ad}(x)$ is multiplication by $\lambda \in \mathbb{Q}_p^\times$. The first assertion is obvious from this. The second assertion follows from the formula $\Delta(x) = s(x)s(x^{-1})^{-1}$, see [11], Corollary 1 to Theorem 2. □

Now let $q : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ denote the canonical quotient morphism, and set

$$G := (\mathbb{Q}_p \times \mathbb{Q}_p/\mathbb{Z}_p) \rtimes \langle \alpha, \beta \rangle,$$

where $\alpha, \beta \in \mathrm{Aut}(\mathbb{Q}_p \times \mathbb{Q}_p/\mathbb{Z}_p)$ are defined by

$$\alpha(x, y) := (x, y + q(x)) \quad \text{and} \quad \beta(x, y) := (px, y),$$

respectively. We give $\langle \alpha, \beta \rangle \leq \mathrm{Aut}(\mathbb{Q}_p \times \mathbb{Q}_p/\mathbb{Z}_p)$ the discrete topology, and we give G the product topology. Then G is a locally compact group and is a one-dimensional p-adic Lie group indeed, and so is its subgroup $H := \ker \Delta_G$. Note that since H is an open subgroup of G, the scale function s_H of H is the restriction of s_G to H.[5] Therefore $s_H \equiv 1$ by Lemma 6.1. For ease of notation we identify $\mathbb{Q}_p$, $\mathbb{Q}_p \times \mathbb{Q}_p/\mathbb{Z}_p$ and $\langle \alpha, \beta \rangle$ with the subgroups $\mathbb{Q}_p \times \{(0, 1)\}$, $\mathbb{Q}_p \times \mathbb{Q}_p/\mathbb{Z}_p \times \{1\}$ and $\{(0, 0)\} \times \langle \alpha, \beta \rangle$, respectively, of G.

Let N be an open, normal subgroup of H – we show that N is not compact. To this end, note that $\mathbb{Q}_p \times \mathbb{Q}_p/\mathbb{Z}_p \subseteq H$, since this is an open, abelian subgroup of G. Furthermore, we have $\alpha \in H$, since α normalizes the open, compact subgroup $\mathbb{Z}_p$ of G. The scale function is a class function; therefore $\beta^n \alpha \beta^{-n} \in H$, for every $n \in \mathbb{Z}$. Now $N \cap \mathbb{Q}_p$ is an open 0-neighbourhood in $\mathbb{Q}_p$, whence there exists $k \in \mathbb{N}$ such that $p^k \mathbb{Z}_p \subseteq N$. Since $(\beta^n \alpha \beta^{-n})(p^k, 0) = (p^k, q(p^{k-n}))$, we conclude that $\{p^k\} \times \mathbb{Q}_p/\mathbb{Z}_p \subseteq N$. The latter subset of N is closed but not compact. Hence N cannot be compact.

Conclusion. H is a uniscalar, 1-dimensional p-adic Lie group which does not have an open, compact, normal subgroup.

7 Applications

Theorem 5.2 is of interest in connection with structural investigations of compactly generated, locally compact groups which are not necessarily p-adic Lie groups.

[5] It is obvious from the definition of tidiness given in [11] that every subgroup of H which is tidy for some $x \in H$ is also tidy for x as a subgroup of G.

If G is a compactly generated, locally compact, totally disconnected group, we let $\mathbb{P}(G)$ denote the set of primes occurring in the prime factor decompositions of the integers $s_G(x)$, where x ranges through G. Then $\mathbb{P}(G)$ is a finite set ([14], Theorem 3.4).

Proposition 7.1. *Suppose that G is a compactly generated, locally compact group, and $f : G \to H$ a continuous homomorphism into a p-adic Lie group H. Let G_1 denote the identity component of G. If $p \notin \mathbb{P}(G/G_1)$, then, for every identity neighbourhood U in G, there exists an open, normal subgroup N of G such that* $\ker f \subseteq N \subseteq U \cdot \ker f$.

Proof. Let $q : G \to G/\ker f =: Q$ denote the canonical quotient morphism, and $f' : Q \to H$ the morphism determined by $f' \circ q = f$. Since Q is locally compact and f' is injective, Q is a p-adic Lie group by [2], §8.2, Corollary 1 to Theorem 2. We may therefore assume w.l.o.g. that f is a quotient morphism. Then H is a compactly generated p-adic Lie group; by Theorem 2.1, $\operatorname{im} s_H \subseteq p^{\mathbb{N}_0}$. Set $G' := G/G_1$, and let $q' : G \to G'$ be the canonical quotient morphism. Since H is totally disconnected, we have $G_1 \leq \ker f$, and there is a unique quotient morphism $g : G' \to H$ such that $g \circ q' = f$. Now g being a quotient morphism, $s_H(g(x))$ divides $s_{G'}(x)$, for every $x \in G'$, see [13], Proposition 4.7. Hence if $p \notin \mathbb{P}(G')$, then H is uniscalar and therefore pro-discrete by Theorem 5.2. Let U be any identity neighbourhood in G. Since $f(U)$ is an identity neighbourhood in H and H is pro-discrete, H has an open, normal subgroup $N' \subseteq f(U)$; then $N := f^{-1}(N')$ has the required properties. □

We presently deduce:

Corollary 7.2. *Suppose that G is a compactly generated, locally compact group, $x \in G$, and suppose that $f : G \to H$ is a continuous homomorphism into a p-adic Lie group such that $f(x) \neq 1$. If $p \notin \mathbb{P}(G/G_1)$, then there exists a continuous homomorphism $g : G \to D$ into a discrete group D such that $g(x) \neq 1$.* □

We conclude this article with results concerning projective limits of p-adic Lie groups. The following lemma is a special case of [13], Proposition 5.4:

Lemma 7.3. *Let G be a pro-p-adic Lie group. Let $\mathcal{N}$ be the set of all closed, normal subgroups N of G such that $G_N := G/N$ is a p-adic Lie group; direct $\mathcal{N}$ via inverse inclusion. Given $N \in \mathcal{N}$, let $q_N : G \to G_N$ be the canonical quotient map. Then*

$$s_G(x) = \lim_{N \in \mathcal{N}} s_{G_N}(q_N(x))$$

for all $x \in G$. In particular, $\operatorname{im} s_G \subseteq p^{\mathbb{N}_0}$. □

Proposition 7.4. *Suppose that G is a compactly generated, pro-p-adic Lie group. If G is uniscalar, then G is pro-discrete; otherwise, $\mathbb{P}(G) = \{p\}$.*

Proof. If G is not uniscalar, then $\{1\} \neq \operatorname{im} s_G \subseteq p^{\mathbb{N}_0}$ by Lemma 7.3: thus $\mathbb{P}(G) = \{p\}$. Now suppose that G is uniscalar, and let U be an identity neighbourhood in G; we pick a compact identity neighbourhood V such that $VV \subseteq U$. Since G is a pro-p-adic Lie group, there exists a closed normal subgroup N of G such that $N \subseteq V$ and $Q := G/N$ is a p-adic Lie group. Let $q : G \to Q$ be the canonical quotient map. Since $s_Q(q(x))$ divides $s_G(x)$ for all $x \in G$ by [13], Proposition 4.7, we deduce that Q is uniscalar. By Theorem 5.2, Q is pro-discrete; hence there exists an open, compact, normal subgroup W of Q such that $W \subseteq q(V)$. Then $q^{-1}(W) \subseteq VN \subseteq VV \subseteq U$ is an open, compact, normal subgroup of G which is contained in U. We deduce that G is pro-discrete. □

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