Extremal Networks and Connectivity

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Certificate of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying subject to the provisions of the Copyright Act 1968.

(Signed) ____

Kim Marshall

Dedication

To the memory of my paternal grandmother, Margaret Isabel Marshall,
(16th August 1910 - 6th February 2005).
Grandma removed my father from school at the age of 14, due to the conviction that formal education was a waste of time. Let us hope that she is wrong on that point.

To the memory of my maternal grandfather, John Francis Lynch,
(13th April 1918 - 29th October, 1993).
Pop beamed with pride at the graduation of his first grandchild, my brother. His departure prompted my return to my home, my country and my studies.

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List of Publications

Publications Arising from This Thesis

1. C. Balbuena, J. Tang, K. Marshall, Y. Lin, Superconnectivity of regular graphs with small diameter, *Discrete Applied Mathematics* **157** (2009), pp. 1349- 1353.
2. C. Balbuena, K. Marshall, L.P. Montejano, On the connectivity and superconnected graphs with small diameter, *Discrete Applied Mathematics* **158** (2010), pp. 397-403.
3. K. Marshall, M. Miller and J. Ryan, Extremal Graphs without Cycles of length 8 or less *Eurocomb 2011*
4. C. Delorme, E. Flandrin, Y. Lin, K. Marshall, M. Miller and J. Ryan, Extremal graphs with small girth, *In preparation.*
5. K. Marshall, M. Miller and J. Ryan, On maximum size of graphs with girth greater than 8, *In preparation.*

Further Publications Produced During my Candidature

6. K. Marshall, J. Ryan, Mode and antimode graphs, *Proceedings of the Sixteenth Australasian Workshop on Combinatorial Algorithms*, (2005), pp. 231- 238.
7. K. Marshall, J. Ryan, On antimode graphs, *The Journal of Combinatorial Mathematics and Combinatorial Computing*, (May 2008), pp. 51-60.
8. K. Marshall, J. Ryan, The eccentric nature of generalised Petersen graphs, *In preparation.*

Abstract

In 1736, in the town of Königsberg it was asked, "Is it possible to walk across the seven bridges that span the river Pregel, which divide the town of Königsberg into four land masses, without having to cross any bridge more than once?" Euler observed that it was unnecessary to consider the size of the land masses, the length of the bridges or the route taken to traverse the bridges in order to answer this question. He showed that the problem could be abstracted by considering only the *topology of the network*, where a network is a group or system of interconnected things or nodes: in this case the four land masses; and a topology is the way in which constituent parts are interrelated or linked: in this case, whether there is a bridge between any two chosen landmasses or not. The Königsberg bridge problem was the first problem in recorded history to be formulated in graph theoretic terms, that is, as the topology of a network. Euler's solution to this problem is considered to be the first theorem of graph theory.

Today, 275 years later, graph theory is a vibrant field of research with remarkably diverse applications, including: molecular chemistry; developing vaccination strategies to prevent the spread of viruses through human populations and computer networks; modelling complex ecological systems; analysis of social networks; and the design of VLSI (very large scale integrated circuits) of multiprocessors. In this thesis we consider questions in two separate but related research areas in the field of graph theory, namely, extremal graph theory and connectivity.

Extremal graph theory is the study of graphs that are extremal, that is, maximal or minimal, under some given constraints. In this thesis we focus on the problem of finding the maximum number of pair-wise connections between the nodes in a network, given the number of nodes and the length of the shortest cycle in the network. A graph that attains this bound is called an extremal graph. Our interest in extremal graphs arose from the problem of determining the structure of the most efficient and reliable networks. We provide constructions that produce infinite families of extremal graphs. We examine the relationship between extremal graphs and some other graphs that have been considered in the design of optimal networks. We develop an algorithm that we use to establish new and improved lower bounds on the size of some extremal graphs and determine the exact size of the extremal graphs for some particular parameters.

A graph is connected if there is a path, consisting of nodes and links, between any two nodes in the graph. The ability to send and receive email via the Internet is dependent upon the Internet being connected, that is, there is a path of computers and connections between the sender and receiver of the email. The connectivity of a network is the number of nodes or links that must be removed in order to partition the network into two or more components. High connectivity of a network corresponds to the properties of fault tolerance and resilience under attack. In this thesis we determine a number of sufficient conditions that ensure good connectivity of a network.

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