On Sampled-Data Models for Nonlinear Systems

Juan I. Yuz, *Student Member, IEEE,* and Graham C. Goodwin, *Fellow, IEEE*

***Abstract*—Models for deterministic continuous-time nonlinear systems typically take the form of ordinary differential equations. To utilize these models in practice invariably requires discretization. In this paper, we show how an approximate sampled-data model can be obtained for deterministic nonlinear systems such that the local truncation error between the output of this model and the true system is of order Δ^{r+1}, where Δ is the sampling period and r is the system relative degree. The resulting model includes extra zero dynamics which have no counterpart in the underlying continuous-time system. The ideas presented here generalize well-known results for the linear case. We also explore the implications of these results in nonlinear system identification.**

***Index Terms*—Nonlinear systems, sampled-data models, sampling zeros, system identification, zero dynamics.**

I. Introduction

MODELS for continuous-time nonlinear systems often arise from the application of physical laws such as conservation of momentum, energy, etc. [1]. These models typically take the form of ordinary differential equations. To utilize these models in a numerical context requires discretization. This raises the question of the relationship between the model describing the samples and the original continuous-time model. It is tempting to simply sample quickly and then to replace derivatives in the continuous-time model by divided differences in the sampled-data model. However, one can obtain a more accurate model. For example, in the linear case, it is well known that better sampled-data models can be generated by including extra zeros due to the sampling process [2].

For linear systems, the presence of sampling zeros has been discussed in many papers following [2]. In that work, it is shown that the sampled-data model corresponding to a linear system of relative degree r has, generically, $r-1$ *sampling zeros*, which have no continuous-time counterpart. When using shift operator models, these sampling zeros converge (in the z-domain) asymptotically, as the sampling period goes to zero, to the roots of the Euler–Fröbenious polynomials [2]–[4]. Equivalent convergence results hold when using the delta operator [5], [6], however, in this case the sampling zeros go to infinity (in the γ-domain). The presence of sampling zeros in stochastic models has also been addressed in [7]–[9].

Sampling zeros are known to have an effect mainly at high frequencies. Nonetheless, they have important consequences in both estimation and control. For example, in the least squares parameter estimation of continuous-time autoregressive models it has been shown that they have to be considered to obtain unbiased parameter estimates [10], [11].

One would reasonably expect similar results to hold for nonlinear systems. However, the situation for the nonlinear case is more complex than for linear systems. Indeed, to the best of our knowledge, an explicit characterization of the *sampling zero dynamics* for nonlinear systems has previously remained unresolved, although, an implicit characterization has been given in [12].

The occurrence of nonlinear zero dynamics is relevant to the problem of control of nonlinear continuous-time systems. In this context, topics such as relative degree, normal form, and zero dynamics of the continuous-time nonlinear plant become important, in particular, regarding feedback linearization techniques [13]–[17]. Some of these results have also been extended to discrete-time and sampled nonlinear systems [18]–[30]. However, the theory for the discrete-time case is less well developed than for the continuous-time case [31] and the absence of good models for sampled-data nonlinear plants is still recognized as an important issue for control design [32]. The accuracy of the approximate sampled-data plant model has proven to be a key issue in the context of control design, where a controller designed to stabilise an approximate model may fail to stabilises the exact discrete-time model, no matter how small the sampling period Δ is chosen [33]. Any sampled-data model for a nonlinear system will, in general, be an approximation of the combination of two elements: the continuous-time system and the associated sample and hold device. An exact discrete-time description of such a hybrid nonlinear system is, in most cases, not known or impossible to compute [34].

In this paper, we present an approximate sampled-data model for nonlinear system which is *accurate* to some order in the sampling period. We show how a particular strategy can be used to approximate the system output and its derivatives in such a way as to obtain a local truncation error, between the output of the resulting sampled-data model and the true continuous-time system output, of order Δ^{r+1}, where Δ is the sampling period and r is the (nonlinear) relative degree. An insightful interpretation of the obtained sampled-data model can be made in terms of additional zero dynamics, which have no continuous-time counterpart. We give an explicit characterization of these *sampling zero dynamics* and show that these are a function only of the system relative degree r. Moreover, the sampling zero dynamics turn out to be identical to those found in the linear case. Thus, the current paper extends the well-known notion of sampling zeros from the linear case to nonlinear systems. We also examine the implications of including these sampling zero dynamics in discrete-time nonlinear models used for system identification.

The remainder of the paper is structured as follows. In Section II, we review known results for sampled linear systems,

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The authors are with the Centre for Complex Dynamic Systems and Control (CDSC), School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan, NSW 2308, Australia (e-mail: juan.yuz@newcastle.edu.au; graham.goodwin@newcastle.edu.au).

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using the delta operator. Concepts and properties of nonlinear systems are presented in Section III. In Section IV, the main result of this paper is presented, namely, a sampled-data model for nonlinear systems. Section V explores the implications of the use of the resultant model in nonlinear system parameter estimation. Finally, conclusions are presented in Section VI.

II. Review of the Linear Case

To set the results of the current paper in context, we begin by reviewing well-known results for sampled linear systems. For convenience we express the results using the delta operator [5], [6]. This formulation will also prove useful in the nonlinear case studied in Section IV. Corresponding results hold for the shift operator using the following relations in discrete-time and complex variable domains:

$$\delta = \frac{q-1}{\Delta} \iff \gamma = \frac{z-1}{\Delta}. \tag{1}$$

We are interested in the sampled-data model for linear systems when the input is a piecewise constant signal generated by a zero-order holdZOH. Thus, for a sampling period Δ

$$u(t) = u(k\Delta) = u_k; \quad k\Delta \leq t < k\Delta + \Delta. \tag{2}$$

We then have the following result.

Lemma 1: Given a sampling period Δ, the exact discrete-time sampled-data model corresponding to the nth order integrator $G(s) = s^{-n}$, $n \geq 1$, for a ZOH input, is given by

$$G_\delta(\gamma) = \frac{p_n(\Delta\gamma)}{\gamma^n} \tag{3}$$

where the polynomial $p_n(\Delta\gamma)$ is given by

$$p_n(\Delta\gamma) = \det M_n \tag{4}$$

and where the matrix M_n is defined by

$$M_n = \begin{bmatrix} 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-2}}{(n-1)!} & \frac{\Delta^{n-1}}{n!} \\ -\gamma & 1 & \dots & \frac{\Delta^{n-3}}{(n-2)!} & \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & -\gamma & 1 & \frac{\Delta}{2!} \\ 0 & \dots & 0 & -\gamma & 1 \end{bmatrix}. \tag{5}$$

Proof: See Appendix I. □

Remark 1: The polynomials $p_n(\Delta\gamma)$ in Lemma 1, when rewritten in terms of the z-variable using (1), correspond to the Euler–Fröbenius polynomials [4]. The role of these polynomials in describing pulse transfer function zeros for linear systems was first described in [2].

Remark 2: In Lemma 1, we have expressed the Euler–Fröbenius polynomials in terms of the delta transform variable γ. However, the definition of these polynomials as the determinant of matrix (5) seems to be novel and differs from the usual format given in the literature [5], [6].

A consequence of Lemma 1 is a recursive relation for the polynomials $p_n(\Delta\gamma)$ described here.

Lemma 2: The polynomials $p_n(\Delta\gamma)$ defined by (4) and (5) satisfy the recursion

$$p_0(\Delta\gamma) \triangleq 1 \tag{6}$$

$$p_n(\Delta\gamma) = \sum_{\ell=1}^{n} \frac{(\Delta\gamma)^{\ell-1}}{\ell!} p_{n-\ell}(\Delta\gamma), \qquad n \geq 1 \tag{7}$$

and

$$\lim_{\Delta\to 0} p_n(\Delta\gamma) = 1 \qquad \forall n \in \{1, 2, \dots\}. \tag{8}$$

Proof: See Appendix II. □

We next consider the case of a general single-input–single-output (SISO) linear continuous-time system. Again, we are interested in the corresponding discrete-time model when a ZOH input is applied. The relationship between the continuous-time poles and those of the discrete-time model can be easily determined. However, the relationship between the zeros in the continuous and discrete domains is much more involved. We consider the asymptotic case as the sampling rate increases.

Lemma 3: Consider an SISO linear continuous-time system described in transfer function form by

$$G(s) = \frac{B(s)}{A(s)} = \frac{K\prod_{\ell=1}^{m}(s-\beta_\ell)}{\prod_{\ell=1}^{n}(s-\alpha_\ell)}, \qquad m < n. \tag{9}$$

Given a sampling period Δ, the discrete-time model corresponding to this system, for a ZOH input, is given by

$$G_\delta(\gamma) = \frac{B_\delta(\gamma)}{A_\delta(\gamma)} \tag{10}$$

where, as the sampling period Δ goes to zero

$$A_\delta(\gamma) = \prod_{\ell=1}^{n}\left(\gamma - \frac{e^{\alpha_\ell\Delta}-1}{\Delta}\right) \longrightarrow A(\gamma) \tag{11}$$

$$B_\delta(\gamma) \longrightarrow B(\gamma)p_{n-m}(\Delta\gamma). \tag{12}$$

Proof: See [2], [3], [5], or [6]. □

III. Nonlinear System

In this section, we review some concepts and results from nonlinear system theory that will be used later in Section IV. The results presented here are based on [13], for continuous-time systems, and partially based on [12] and [25], for the discrete-time case.

A. Continuous-Time Systems

Much of the existing work regarding control of (continuous-time) nonlinear systems utilizes a model consisting of a set of ordinary differential equations affine in the control signals [13]

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{13}$$

$$y(t) = h(x(t)) \tag{14}$$

where $x(t)$ is the state evolving in an open subset $\mathcal{M} \subset \mathbb{R}^n$, and where the vector fields $f(\cdot)$ and $g(\cdot)$, and the output function $h(\cdot)$ are analytic.

Definition 1 (Relative Degree): The nonlinear system (13)–(14) is said to have relative degree r at a point x_o if

i) $L_g L_f^k h(x) = 0$ for x in a neighborhood of x_o and for $k = 0, \ldots, r-2$;
ii) $L_g L_f^{r-1} h(x_o) \neq 0$;

where L_g and L_f correspond to Lie derivatives [13]. For example, $L_g h(x) = (\partial h/\partial x) g(x)$.

Intuitively, the relative degree, as defined previously, corresponds to the number of times that one needs to differentiate the output $y(t)$ to make the input $u(t)$ appear explicitly.

We next show that there is a local coordinate transformation that allows one to rewrite the nonlinear system (13)–(14) in the so called *normal form*.

Lemma 4 (Local Coordinate Transformation): Suppose that the system has relative degree r at x_o. Consider the new system coordinates defined as

$$z_1 = \phi_1(x) = h(x) \tag{15}$$
$$z_2 = \phi_2(x) = L_f h(x) \tag{16}$$
$$\vdots$$
$$z_r = \phi_r(x) = L_f^{r-1} h(x). \tag{17}$$

Furthermore, if $r < n$ it is always possible to define $z_{r+1} = \phi_{r+1}(x), \ldots, z_n = \phi_n(x)$ such that

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \Phi(x) \tag{18}$$

has a nonsingular Jacobian at x_o. Then, $\Phi(\cdot)$ is a local coordinate transformation in a neighborhood of x_o. Moreover, it is always possible to define $z_{r+1} = \phi_{r+1}(x), \ldots, z_n = \phi_n(x)$ in such a way that

$$L_g \phi_i(x) = 0 \tag{19}$$

in a neighborhood of x_o, for all $i = r+1, \ldots, n$.

Proof: See [13]. □

Lemma 5 (Normal Form): The state–space description of the nonlinear system (13)–(14) in the new system coordinates defined by Lemma 4 is given by the so-called normal form

$$\dot{\zeta} = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{r-1} & \\ 0 & & & \\ \hline 0 & 0 & \ldots & 0 \end{array}\right] \zeta + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (b(\zeta,\eta) + a(\zeta,\eta)u(t)) \tag{20}$$

$$\dot{\eta} = c(\zeta, \eta) \tag{21}$$

where the output is $z_1 = h(x) = y$, the state vector is

$$z(t) = \begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix} \quad \begin{cases} \zeta(t) = [z_1(t), z_2(t), \ldots, z_r(t)]^T \\ \eta(t) = [z_{r+1}(t), z_{r+2}(t), \ldots, z_n(t)]^T \end{cases} \tag{22}$$

and

$$b(\zeta,\eta) = b(z) = L_f^r h(\Phi^{-1}(z)) \tag{23}$$

$$a(\zeta,\eta) = a(z) = L_g L_f^{r-1} h(\Phi^{-1}(z)) \tag{24}$$

$$c(\zeta,\eta) = c(z) = \begin{bmatrix} L_f \phi_{r+1}(\Phi^{-1}(z)) \\ \vdots \\ L_f \phi_n(\Phi^{-1}(z)) \end{bmatrix} \tag{25}$$

Proof: See [13]. □

Remark 3: Note that the state variables contained in $\zeta(t)$, defined in (15)–(17), correspond to the output $y(t)$ and its first $r-1$ derivatives

$$z_\ell(t) = z_1^{(\ell-1)}(t) = y^{(\ell-1)}(t), \qquad \ell = 1, \ldots, r. \tag{26}$$

Definition 2 (Zero Dynamics): The zero dynamics of the nonlinear system (13)–(14) are defined as the internal dynamics that appear in the system when the input and initial conditions are chosen in such a way as to make the output identically zero, i.e., $y(t) = 0, \forall t$.

Using the coordinate transformation, and, thus, the system expressed in the normal form (20)–(21), we can see that the zero dynamics satisfy

$$\dot{\eta} = c(0, \eta) \tag{27}$$

for any initial condition $z(0) = [0, \eta(0)^T]^T$, and, from (20), for an input

$$u(t) = u_{zd}(t) = -\frac{b(0,\eta)}{a(0,\eta).} \tag{28}$$

Remark 4: For linear systems, the zero dynamics correspond to the system zeros. In this case, (27) reduces to a linear differential equation $\dot{\eta} = S\eta$, where the eigenvalues of the matrix S are the roots of the polynomial $B(s)$ in (19) (see, for example, [13]).

B. Discrete-Time Systems

In this section, we consider the case of nonlinear systems defined in discrete-time. We summarize, in a similar fashion to the aforementioned continuous-time case, several concepts and results partially based on [12] and [25]. See also related work in [35].

We consider the class of nonlinear discrete-time system expressed as

$$\delta x_k = F(x_k) + G(x_k) u_k \tag{29}$$
$$y_k = H(x_k) \tag{30}$$

where $F(\cdot), G(\cdot)$, and $H(\cdot)$ are assumed analytic. Note that the state (29) can also be easily rewritten using the shift operator

$$q x_k = x_{k+1} = F_q(x_k) + G_q(x_k) u_k \tag{31}$$

where, using (1)

$$F_q(x_k) = x_k + \Delta F(x_k) \quad \text{and} \quad G_q(x_k) = \Delta G(x_k). \tag{32}$$

Definition 3 (Discrete-Time Relative Degree): The discrete-time system (29)–(30) has relative degree r if [25]

i) $(\partial y_{k+\ell}/\partial u_k)|_{(x_k,u_k)} = 0$, for all $\ell = 0, \ldots, r-1$;
ii) $(\partial y_{k+r}/\partial u_k)|_{(x_k,u_k)} \neq 0$.

Intuitively, the discrete-time relative degree corresponds to the number of time shifts before an element u_k of the input sequence u appears explicitly in the output sequence y. The relative degree r can also be characterized in terms of divided differences of y_k, as follows.

Lemma 6: The conditions in Definition 3 are equivalent to

a) $(\partial(\delta^\ell y_k))/(\partial u_k)|_{(x_k,u_k)} = 0$, for all $\ell = 0,\ldots,r-1$;
b) $(\partial(\delta^r y_k))/(\partial u_k)|_{(x_k,u_k)} \neq 0$.

Proof: See Appendix III. □

Definition 4 (Discrete-Time Normal Form): Consider the nonlinear discrete-time system (29)–(30) and assume that it has relative degree r. We say that the system is expressed in its discrete-time normal form when it is rewritten as

$$\delta\zeta_k = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{r-1} & \\ 0 & & & \\ \hline 0 & 0 & \ldots & 0 \end{array}\right]\zeta_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}(B(z_k) + A(z_k)u_k) \tag{33}$$

$$\delta\eta_k = C(z_k) \tag{34}$$

where the state vector is

$$z_k = \begin{bmatrix} \zeta_k \\ \eta_k \end{bmatrix} \quad \begin{cases} \zeta_k = [z_{1,k}, z_{2,k}, \ldots, z_{r,k}]^T \\ \eta_k = [z_{r+1,k}, z_{r+2,k}, \ldots, z_{n,k}]^T \end{cases} \tag{35}$$

and the output is $z_{1,k} = H(x_k) = y_k$.

Remark 5: The state variables contained in ζ_k, defined in (35), correspond, in fact, to y_k and its first $r-1$ divided differences, i.e.,

$$z_{\ell,k} = \delta^{\ell-1} z_{1,k} = \delta^{\ell-1} y_k \qquad \forall \ell = 1,\ldots,r. \tag{36}$$

Definition 5 (Discrete-Time Zero Dynamics): The discrete-time zero dynamics of the nonlinear system (29)–(30) are defined as the internal dynamics that appear in the system when the input and initial conditions are chosen in such a way as to make the output identically zero, i.e., $y_k = 0, \forall k$.

If the system is expressed in the normal form (33)–(34), we can see that the zero dynamics satisfy

$$\delta\eta_k = C(0,\eta_k) \tag{37}$$

for any initial condition $z_0 = [0, \eta_0^T]^T$, and, from (33), for an input

$$u_k = u_k^{zd} = -\frac{B(0,\eta_k)}{A(0,\eta_k).} \tag{38}$$

Remark 6: Similarly to the continuous-time case in Remark 4, when restricting ourselves to linear systems, the discrete-time zero dynamics (37) reduce to a linear difference equation $\delta\eta = S\eta$, where the eigenvalues of the matrix S correspond to the zeros of the discrete-time transfer function (10).

The following lemma re-establishes Lemma 1 regarding the sampled model for an nth-order integrator. We show, via use of the normal form, that the eigenvalues of the zero dynamics in this case correspond to the *sampling zeros* of the discrete-time transfer function (3). The latter result will be used for the nonlinear case as a key building block in the Proof of Theorem 2 in Section IV.

Lemma 7 (Sampled nth Order Integrator in Normal Form): Given a sampling period Δ, the discrete-time sampled-data model corresponding to the nth order integrator $G(s) = s^{-n}, n \geq 1$, for a ZOH input, can be written in the normal form

$$\delta z_1 = q_{11} z_1 + Q_{12}\eta + \frac{\Delta^{n-1}}{n!} u_k \tag{39}$$

$$\delta\eta = Q_{21} z_1 + Q_{22}\eta \tag{40}$$

with output $y = z_1$. The scalar q_{11} and the matrices Q_{12}, Q_{21}, and Q_{22} take specific forms as given in (127) in Appendix IV. Furthermore, the sampling zeros in (3) appear as eigenvalues of the matrix Q_{22}, i.e.,

$$p_n(\Delta\gamma) = \det M_n = \frac{\Delta^{n-1}}{n!}\det(\gamma I_{n-1} - Q_{22}). \tag{41}$$

Proof: See Appendix IV. □

IV. Sampled-Data Model for Nonlinear Systems

In this section, we present the main result of this paper, namely, a sampled-data model that approximates the input-output mapping of a given nonlinear system. We also show that this discrete-time model contains extra *zero dynamics* which are the same as the dynamics associated with the asymptotic sampling zeros in the linear case.

We are interested in obtaining a discrete-time model that closely approximates the nonlinear input-output mapping given by (13)–(14), when the input $u(t)$ is generated by a digital device using a ZOH. This will result in a model of the form

$$\delta x^S = f^S(x^S) + g^S(x^S)u \tag{42}$$

$$y^S = h^S(x^S) \tag{43}$$

where $x^S = x_k^S \in \mathbb{R}^n$ is the discrete-time state sequence, $u = u_k$ is the input sequence, $y^S = y_k^S$ is the output sequence, and $k \in \mathbf{Z}$ is the discrete-time index.

Our goal is to define the discrete-time model (42)–(43), such that y^S is *close* (in a well defined sense) to the continuous-time output $y(t)$ in (14) at the sampling instants $t = k\Delta$, when the input $u(t)$ is generated from u_k with the ZOH (2). Theorem 1 explicitly defines the vector fields $f^S(\cdot), g^S(\cdot)$, and $h^S(\cdot)$ in (42) and (43) in terms of the sampling period Δ and the vector fields $a(\cdot), b(\cdot)$, and $c(\cdot)$ in Lemma 5, which are function of $f(\cdot), g(\cdot)$, and $h(\cdot)$ in the original continuous-time nonlinear model (13)–(14). We first introduce the following assumption.

Assumption 1: The continuous-time nonlinear system (13)–(14) has uniform relative degree $r \leq n$ in the open subset $\mathcal{M} \subset \mathbb{R}^n$, where the state $x(t)$ evolves.

This assumption ensures that there is a coordinate transformation as in Lemma 4 that allows us to express the system in its normal form. We then have the following key result.

Theorem 1: Consider the continuous-time nonlinear system (13)–(14) subject to Assumption 1. Then the local truncation error between the output $y^S = z_1^S$ of the following discrete-time nonlinear model and the true system output $y(t)$ is of order Δ^{r+1}:

$$\delta\zeta^S = \begin{bmatrix} 0 & 1 & \frac{\Delta}{2} & \cdots & \frac{\Delta^{r-2}}{(r-1)!} \\ 0 & 0 & 1 & \cdots & \frac{\Delta^{r-3}}{(r-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & 0 & & \dots & 0 \end{bmatrix} \zeta^S + \begin{bmatrix} \frac{\Delta^{r-1}}{r!} \\ \frac{\Delta^{r-2}}{(r-1)!} \\ \vdots \\ \frac{\Delta}{2} \\ 1 \end{bmatrix} (b + au) \tag{44}$$

$$\delta\eta^S = c(\zeta^S, \eta^S) \tag{45}$$

where $a = a(\zeta^S, \eta^S), b = b(\zeta^S, \eta^S)$, and $c(\zeta^S, \eta^S)$ are defined in Lemma 5, u is the discrete-time input to the ZOH, and the discrete-time state vector is

$$z^S = \begin{bmatrix} \zeta^S \\ \eta^S \end{bmatrix} \quad \begin{cases} \zeta^S = [z_1^S, z_2^S, \dots, z_r^S]^T \\ \eta^S = [z_{r+1}^S, z_{r+2}^S, \dots, z_n^S]^T. \end{cases} \tag{46}$$

Proof: Assumption 1 ensures the existence of the normal form for the nonlinear model (13)–(14). In Lemma 5, the vector fields $b(\cdot)$, $a(\cdot)$, and $c(\cdot)$ are continuous and, thus, the state variables $z_1(t), \dots, z_r(t)$ are continuous functions of t. This implies (see Remark 3) that the output signal $y(t)$ and its first $r-1$ derivatives are continuous. However, when the input signal $u(t)$ is generated by a ZOH, the rth derivative, $y^{(r)}(t) = \dot{z}_r(t) = b(z) + a(z)u(t)$, is well defined but is, in general, discontinuous at the sampling instants $t = k\Delta$, when the control signal (2) is updated. This allows us to apply the *Taylor's formula with remainder* [36, Th. 5.19] to $y(t)$ and to each one of its $r-1$ derivatives at any point t_o as

$$y(t_o + \tau) = y(t_o) + y^{(1)}(t_o)\tau + \cdots + \frac{y^{(r)}(\xi_1)}{r!}\tau^r \tag{47}$$

$$y^{(1)}(t_o + \tau) = y^{(1)}(t_o) + \cdots + \frac{y^{(r)}(\xi_2)}{(r-1)!}\tau^{r-1} \tag{48}$$

$$\vdots$$

$$y^{(r-1)}(t_o + \tau) = y^{(r-1)}(t_o) + y^{(r)}(\xi_r)\tau \tag{49}$$

for some $t_o < \xi_\ell < t_o + \tau$, for all $\ell = 1, \dots, r$.

In turn, this implies that, taking $t_o = k\Delta$ and $\tau = \Delta$, the state variables z_ℓ at $t = k\Delta + \Delta$ can be expressed **exactly** by

$$z_1(k\Delta + \Delta) = z_1(k\Delta) + \Delta z_2(k\Delta) + \cdots + \frac{\Delta^r}{r!}[b + a\,u]_{t=\xi_1} \tag{50}$$

$$z_2(k\Delta + \Delta) = z_2(k\Delta) + \cdots + \frac{\Delta^{r-1}}{(r-1)!}[b + a\,u]_{t=\xi_2} \tag{51}$$

$$\vdots$$

$$z_r(k\Delta + \Delta) = z_r(k\Delta) + \Delta\,[b + a\,u]_{t=\xi_r} \tag{52}$$

and

$$\eta(k\Delta + \Delta) = \eta(k\Delta) + \Delta[q]_{t=\xi_{r+1}} \tag{53}$$

for some time instants $k\Delta < \xi_\ell < k\Delta + \Delta, \ell = 1, \dots, r+1$.

Next, we rewrite (50)–(53) using the δ-operator. We also replace the signals at the sampling instants by their sampled counterparts, using the superscript ^{S}

$$\delta z_1^S = z_2^S + \frac{\Delta}{2} z_3^S + \cdots + \frac{\Delta^{r-1}}{r!}[b(\zeta,\eta) + a(\zeta,\eta)u]_{t=\xi_1} \tag{54}$$

$$\delta z_2^S = z_3^S + \cdots + \frac{\Delta^{r-2}}{(r-1)!}[b(\zeta,\eta) + a(\zeta,\eta)u]_{t=\xi_2} \tag{55}$$

$$\vdots$$

$$\delta z_r^S = [b(\zeta,\eta) + a(\zeta,\eta)u]_{t=\xi_r} \tag{56}$$

$$\delta\eta^S = [c(\zeta,\eta)]_{t=\xi_{r+1}}. \tag{57}$$

Note that this is an exact discrete-time description of the nonlinear system together with a ZOH input, for some (undetermined) time instants $\xi_\ell, \ell = 1, \dots, r+1$. Replacing these unknown time instants by $k\Delta$ we obtain the approximate discrete-time model in (44) and (45).

We next analyze the local truncation error [37] between the true system output and the output of the obtained sampled data model, assuming that, at $t = k\Delta$, the state z^S is equal to the true system state $z(k\Delta)$. We compare the true system output at the end of the sampling interval, $y(k\Delta + \Delta) = z_1(k\Delta + \Delta)$ in (50), with the first (shifted) state of the approximate sampled-data model in (44), i.e., with:

$$\begin{aligned} qz_1^S &= (1 + \Delta\delta)\, z_1^S \\ &= z_1^S + \Delta z_2^S + \cdots + \frac{\Delta^r}{r!}[b(\zeta^S,\eta^S) + a(\zeta^S,\eta^S)u]. \end{aligned} \tag{58}$$

This yields the following local truncation output error:

$$\begin{aligned} e &= |y(k\Delta + \Delta) - q\,y^S| \\ &= \frac{\Delta^r}{r!}\,|[b(\zeta,\eta) + a(\zeta,\eta)u_k]_{t=\xi_1} \\ &\quad - [b(\zeta,\eta) + a(\zeta,\eta)u_k]_{t=k\Delta}| \\ &\le \frac{\Delta^r}{r!} \cdot L\|(\zeta,\eta)_{t=\xi_1} - (\zeta,\eta)_{t=k\Delta}\| \\ &= \frac{\Delta^r}{r!} \cdot L\|z(\xi_1) - z(k\Delta)\| \end{aligned} \tag{59}$$

where the existence of the Lipschitz constant $L > 0$ is guaranteed by the analyticity of $f(\cdot), g(\cdot)$, and $h(\cdot)$ in (13)–(14) and, as a consequence, of $a(\cdot), b(\cdot)$, and $c(\cdot)$. Indeed, any C^1 map satisfies locally a Lipschitz condition at each point [38].

Furthermore, according to [37, Th. 112E], the Lipschitz condition guarantees that the variation of the state trajectory $z(t)$ can be bounded as

$$\begin{aligned} \|z(\xi_1) - z(k\Delta)\| &\le C \cdot \frac{e^{L|\xi_1 - k\Delta|} - 1}{L} \\ &< C \cdot \frac{e^{L\Delta} - 1}{L} = \mathcal{O}(\Delta). \end{aligned} \tag{60}$$

The result then follows from (59). □

Remark 7: The Taylor series truncation used in the proof of Theorem 1 is closely related to Runge–Kutta methods [37], commonly used to simulate nonlinear systems. In fact, the model in Theorem 1 describes an approximate model for the output $y(t)$ and its derivatives to *solve* the nonlinear differential equation in one sampling interval. An important observation that we will explore in Theorem 2 is that this improved numerical integration technique can be interpreted as incorporating sampling zero dynamics into the discrete-time nonlinear model.

Remark 8: Theorem 1 shows that the accuracy of the approximate sampled-data model improves with the continuous-time system relative degree r. Thus, in general, one obtains a more accurate model than the one resulting from simple derivative replacement using an Euler approximation.

Remark 9: The sampled-data model described in Theorem 1 can be obtained for any equivalent representation of the nonlinear system of the form (13)–(14). Specifically, the approximate sampled-data model (44)–(45) is described in terms of $a(\cdot), b(\cdot)$, and $c(\cdot)$ which are functions of $f(\cdot), g(\cdot)$, and $h(\cdot)$ (see Lemma 5).

Remark 10: In [25], a *sampled normal form* is obtained by a Taylor series expansion of all the elements of the state vector (22) to the same order in the sampling period Δ. By way of contrast, we have considered the smoothness of the input $u(t)$, and, thus, of $y(t)$ and its derivatives, to obtain the exact representation given in (54)–(57) and, from there, the approximate sampled-data model (44)–(45).

Remark 11: The result in Theorem 1 can equally be applied to the nonuniform sampling case. In the latter case, the local truncation output error will be of order in Δ_k^{r+1}, where Δ_k is the length of the sampling interval $[t_k, t_{k+1}[$.

Next, we present a result which shows that the discrete-time zero dynamics of the sampled-data model presented in Theorem 1 are given by the sampled counterpart of the continuous-time zero dynamics, together with extra zero dynamics produced by the sampling process. The latter dynamics are linear and, surprisingly, turn out to be the same as those which appear asymptotically for the linear case, as the sampling period goes to zero.

Theorem 2: The sampled-data model (44)–(45) generically has relative degree 1, with respect to the output $z_1^S = y^S$. Furthermore, the discrete-time zero dynamics are given by two subsystems.

i) The sampled counterpart of the continuous-time zero dynamics

$$\delta\eta^S = c\left(0, \tilde{z}_{2:r}^S, \eta^S\right) \tag{61}$$

where $\tilde{z}_{2:r}^S \triangleq [\tilde{z}_2^S, \ldots, \tilde{z}_r^S]^T$.

ii) A linear subsystem of dimension $r-1$

$$\delta\tilde{z}_{2:r}^S = Q_{22}\tilde{z}_{2:r}^S \tag{62}$$

where the eigenvalues of matrix Q_{22} are the same sampling zeros that appear in the asymptotic linear case, namely, the roots of $p_r(\Delta\gamma)$ defined in (4).

Proof: Using the definition of discrete-time relative degree given in Lemma 6, we have that

$$\frac{\partial y^S}{\partial u} = \frac{\partial z_1^S}{\partial u} = 0 \tag{63}$$

$$\begin{aligned}\frac{\partial(\delta y^S)}{\partial u} &= \frac{\partial(\delta z_1^S)}{\partial u} \\ &= \frac{\partial}{\partial u}\left(z_2^S + \cdots + \frac{\Delta^{r-1}}{r!}[b + au]\right) \neq 0\end{aligned} \tag{64}$$

which shows that (44)–(45) has relative degree 1.

Next, in order to extract the zero dynamics of the discrete-time nonlinear system (44)–(45), we rewrite it in its normal form. To do so, we proceed as in the proof of Lemma 7 for the nth order integrator (see Appendix IV). We first define the following linear state transformation:

$$\tilde{\zeta}^S = \begin{bmatrix}\tilde{z}_1^S \\ \vdots \\ \tilde{z}_r^S\end{bmatrix} = T\begin{bmatrix}z_1^S \\ \vdots \\ z_r^S\end{bmatrix} = T\zeta^S \tag{65}$$

where matrix T is defined analogously to (125)

$$T = \left[\begin{array}{c|c}1 & 0 \\ \hline T_{21} & I_{r-1}\end{array}\right] \iff T^{-1} = \left[\begin{array}{c|c}1 & 0 \\ \hline -T_{21} & I_{r-1}\end{array}\right] \tag{66}$$

where

$$T_{21} = -\left[\begin{array}{ccc}\frac{r}{\Delta} & \cdots & \frac{r!}{\Delta^{r-1}}\end{array}\right]^T. \tag{67}$$

Substituting (65)–(66) into (44), we obtain a discrete-time normal form

$$\begin{aligned}\delta\tilde{\zeta}^S = \delta\begin{bmatrix}\tilde{z}_1^S \\ \tilde{z}_{2:r}^S\end{bmatrix} &= \left[\begin{array}{c|c}q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22}\end{array}\right]\begin{bmatrix}\tilde{z}_1^S \\ \tilde{z}_{2:r}^S\end{bmatrix} \\ &\quad + \begin{bmatrix}\frac{\Delta^{n-1}}{n!}(b(\tilde{\zeta}^S, \eta^S) + a(\tilde{\zeta}^S, \eta^S)u) \\ 0\end{bmatrix}\end{aligned} \tag{68}$$

$$\delta\eta^S = c(\tilde{\zeta}^S, \eta^S) = q\left(\tilde{z}_1^S, \tilde{z}_{2:r}^S, \eta^S\right) \tag{69}$$

where the sub-matrices in (68) are given by expressions analogous to (126)–(129) in Appendix IV.

Taking the output $y^S = z_1^S = \tilde{z}_1^S = 0$, for all $k \in \mathbf{Z}$, we now see that the discrete-time zero dynamics are described by two subsystems:

$$\delta\tilde{z}_{2:r}^S = Q_{22}\tilde{z}_{2:r}^S \tag{70}$$

$$\delta\eta^S = q\left(0, \tilde{z}_{2:r}^S, \eta^S\right) \tag{71}$$

and the eigenvalues of Q_{22} are clearly the same as the roots of $p_r(\Delta\gamma)$ as given earlier in Lemma 7. □

Remark 12: If the continuous-time input $u(t)$ is generated by a different hold device, for example, a first-order hold (FOH), this information can be used to include more terms in the Taylor's expansion (50)–(52). This, of course, would lead us to a different approximate discrete-time model in Theorem 1, with different sampling zeros in Theorem 2. In fact, this corresponds to well-known results for the linear case where the asymptotic sampling zeros depend *inter alia* on the nature of the hold device [3], [4], [39], [40].

V. Implications in Nonlinear System Identification

The results given in the previous sections give additional insight to many problems in nonlinear system theory. As a specific illustration, we next consider the problem of nonlinear system identification based on sampled output observations. Note that we do not explicitly consider noise in this paper since our focus is on the deterministic (bias) errors resulting from under-modeling in sampled-data models.

The results in Section IV describe an approximate sampled-data discrete-time model for a nonlinear system. This model shows that the accuracy of the sampled data model can be improved by using a better derivative approximation than simple Euler, where d/dt is replaced by the delta operator δ. This more accurate discrete-time model can be interpreted as including *sampling zero dynamics*, which are the same as in the linear system case.

In this section, we illustrate the use of the approximate sampled-data model (44)–(45) for parameter estimation of a particular nonlinear system. This model, which includes sampling zero dynamics, gives better results than those achieved by simply replacing time derivatives by divided differences, even when fast sampling rates are utilized.

Example 1: Consider the nonlinear system defined by the differential equation

$$\ddot{y}(t) + \alpha_1\dot{y}(t) + \alpha_0 y(t)(1+\varepsilon_1 y^2(t)) = \beta_0(1+\varepsilon_2 y(t))u(t). \quad (72)$$

This model can be expressed in state–space form as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ f(x_1,x_2,u)\end{bmatrix} \quad (73)$$

$$y = x_1 \quad (74)$$

where we have defined the function

$$f(x_1,x_2,u) = -\alpha_1 x_2 - \alpha_0 x_1\left(1+\varepsilon_1 x_1^2\right) + \beta_0(1+\varepsilon_2 x_1)u. \quad (75)$$

This system has relative degree $r=2$ for all $x_o \in \mathbb{R}^2$, and is already in normal form (20)–(21).

The nonlinear function (75) can be linearly reparameterised as $f(x_1,x_2,u) = \phi(t)^T\theta$, where

$$\phi(t) = \begin{bmatrix} -x_2(t) \\ -x_1(t) \\ -x_1(t)^3 \\ u(t) \\ x_1(t)u(t)\end{bmatrix} \quad \text{and} \quad \theta = \begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \alpha_0\\ \varepsilon_1\alpha_0\\ \beta_0\\ \varepsilon_1\beta_0\end{bmatrix}. \quad (76)$$

We next perform system identification by applying an equation error procedure on three different model structures:.

1) A simple derivative replacement model (SDRM): This model is obtained by simply replacing the time derivatives by divided differences in the state–space model (73)–(74). This leads to the approximate model

$$\text{SDRM}: \delta^2 y = -\theta_1\delta y - \theta_2 y - \theta_3 y^3 + \theta_4 u + \theta_5 u y \quad (77)$$

where the parameters θ_i are given in (76).

2) A model incorporating fixed zero dynamics (MIFZD): This is based on our proposed discrete-time nonlinear model in Theorem 1. The corresponding state space representation is given by:

$$\delta x_1 = x_2 + \frac{\Delta}{2} f(x_1,x_2,u) \quad (78)$$

$$\delta x_2 = f(x_1,x_2,u) \quad (79)$$

where $f(x_1,x_2,u)$ is defined in (75). *This particular system* can be rewritten as a divided difference equation as follows:

$$\text{MIFZD}: \delta^2 y = -\theta_1\delta y + \left(1+\frac{\Delta}{2}\delta\right)(-\theta_2 y - \theta_3 y^3 + \theta_4 u + \theta_5 y u) \quad (80)$$

where the parameters θ_i are given in (76).

3) A model incorporating parameterised zero dynamics (MIPZD): This is also based on our proposed discrete-time nonlinear model (78)–(79), with the difference that we expand (80) and relax the existing relation between the parameters of the different terms. This yields

$$\text{MIPZD}: \delta^2 y = -\theta_1\delta y - \theta_2 y - \theta_3 y^3 + \theta_4 u + \theta_5 y u - \theta_6\delta(y^3) + \theta_7\delta u + \theta_8\delta(yu) \quad (81)$$

where $\theta_1 = \alpha_1 + (\Delta/2)\alpha_0$, $\{\theta_2,\ldots,\theta_5\}$ are given in (76), $\theta_6 = (\Delta/2)\alpha_0\varepsilon_1$, $\theta_7 = (\Delta/2)\beta_0$, and $\theta_8 = (\Delta/2)\beta_0\varepsilon_2$.

Note that the MIPZD in (81) can be rewritten in state–space form as

$$\delta x_1 = x_2 - \theta_1 x_1 - \theta_6 x_1^3 + \theta_7 u + \theta_8 u x_1 \quad (82)$$

$$\delta x_2 = -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 u + \theta_5 u x_1 \quad (83)$$

with output $y = x_1$.

The parameters for the three models, SDRM in (77), MIFZD in (80), and MIPZD in (81), can be estimated using the ordinary least squares method by minimizing the *equation error* cost function

$$J_{ee}(\theta) = \frac{1}{N}\sum_{k=0}^{N-1} e_k(\theta)^2 = \frac{1}{N}\sum_{k=0}^{N-1}\left(\delta^2 y - \phi_k^T\theta\right)^2 \quad (84)$$

where (85), as shown at the bottom of the next page, holds.

The parameters for each model were estimated by performing 50 Monte Carlo simulations, using different realizations of a Gaussian random input sequence u_k (zero mean, unit variance). The sampling period was $\Delta = \pi/20$ [s]. The results are summarized in Table I. We can see that both MIFZD and MIPZD give good estimates for the continuous-time parameters, whereas SDRM is not able to find the right values, especially for the parameters $\{\theta_3,\theta_4,\theta_5\}$. Of course, small discrepancies from the continuous-time parameters are explained by the non infinitesimal sampling period.

To explore the convergence of the parameter estimates to the continuous-time values, we repeat the simulations for different sampling periods. Table II shows the root mean square error between the average parameters obtained by running 50 Monte Carlo simulations for each sampling period. Note that we are

TABLE I
PARAMETER ESTIMATES USING EQUATION ERROR PROCEDURES

	CT	SDRM		MIFZD		MIPZD	
		avg	std	avg	std	avg	std
θ_1	3	2.6987	0.4622	2.6479	0.0241	2.6414	0.0141
θ_2	2	1.5080	1.2832	1.5999	0.0487	1.5876	0.0330
θ_3	1	5.8089	30.2745	0.6442	1.0831	0.7299	0.3986
θ_4	2	0.7431	0.1467	1.6054	0.0081	1.5882	0.0052
θ_5	1	0.1597	0.9703	0.7752	0.0557	0.7770	0.0317
θ_6	–	–	–	–	–	0.1152	0.0959
θ_7	–	–	–	–	–	0.1345	0.0004
θ_8	–	–	–	–	–	0.0665	0.0022
$J_{ee}(\theta)$		0.6594	0.1493	0.0069	0.0021	0.0001	0.0001
Validation		0.7203		0.0076		0.0003	

able to compare only the first five parameters of the MIPZD. In fact, we can see that, as the sampling period is reduced this is the model that gives the best estimation of the true parameter vector. On the other hand, the estimate corresponding to the SDRM is clearly asymptotically biased.

We also tested the three models, SDRM, MIFZD, and MIPZD, with the *average estimated parameters* that appear in Table I, using a longer validation data set of length 100 [s] and the same sampling period $\Delta = \pi/20$ [s]. Part of the output of the nonlinear continuous-time system and the discrete-time models, when using the validation input, are shown in Fig. 1. We see that both models based on our proposed state-space model as described in Section IV replicate the continuous-time output very accurately. On the other hand, the SDRM has a clear bias.

The value of the equation error cost function (84) for each one of the three discrete-time models, when considering the sampled input and output validation data, appears in the last row of Table I.

Remark 13: The results obtained for the nonlinear models in the previous example highlight that the inclusion of zero dynamics (MIFZD and MIPZD) allows one to obtain better results than a simple derivative replacement approach (SDRM). In particular, the latter model will give biased estimates also in the linear system case. As a matter of fact, if we consider $\varepsilon_1 = \varepsilon_2 = 0$ in (72) we obtain the linear system

$$\ddot{y}(t) + \alpha_1 \dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t). \tag{86}$$

This system can also be represented by the transfer function

$$G(s) = \frac{\beta_0}{s^2 + \alpha_1 s + \alpha_0}. \tag{87}$$

TABLE II
CONVERGENCE OF PARAMETER ESTIMATES

	$\sqrt{\frac{1}{5}\sum_{i=1}^{5}\left(\bar{\theta}_i - {\theta_i}^{ct}\right)^2}$		
Δ	SDRM	MIFZD	MIPZD
$\pi/20$	2.2691	0.3513	0.3438
$\pi/100$	9.6156	0.0744	0.0714
$\pi/200$	53.6027	0.0508	0.0366
$\pi/500$	109.5187	0.0167	0.0146

If derivatives in (86) are replaced by divided differences, we obtain the lineal SDRM model

$$\delta^2 y + \alpha_1 \delta y + \alpha_0 y = \beta_0 u. \tag{88}$$

The parameter estimates that minimize the equation error cost function

$$J(\hat{\theta}) = \lim_{N\to\infty} \frac{1}{N} \sum_{0}^{N-1} e_k(\hat{\theta})^2 = E\{e_k(\hat{\theta})^2\} \tag{89}$$

where

$$e_k = \delta^2 y + \hat{\alpha}_1 \delta y + \hat{\alpha}_0 y - \hat{\beta}_0 u \tag{90}$$

are given by the solution of the equation

$$\frac{dJ(\hat{\theta})}{d\hat{\theta}} = \nabla J(\hat{\theta}) = 0. \tag{91}$$

Thus, differentiating the cost function with respect to each of the parameter estimates, we obtain

$$\begin{bmatrix} E\{(\delta y)^2\} & E\{(\delta y) y\} & -E\{(\delta y) u\} \\ E\{(\delta y) y\} & E\{y^2\} & -E\{yu\} \\ -E\{y^2\} & -E\{yu\} & E\{u^2\} \end{bmatrix} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_0 \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} -E\{(\delta y)(\delta^2 y)\} \\ -E\{y \delta^2 y\} \\ E\{u \delta^2 y\} \end{bmatrix}. \tag{92}$$

This equation can be rewritten in terms of (discrete-time) correlations as

$$\begin{bmatrix} \frac{2r_y(0)-2r_y(1)}{\Delta^2} & \frac{r_y(1)-r_y(0)}{\Delta} & \frac{r_{yu}(0)-r_{yu}(1)}{\Delta} \\ \frac{r_y(1)-r_y(0)}{\Delta} & r_y(0) & -r_{yu}(0) \\ \frac{r_{yu}(0)-r_{yu}(1)}{\Delta} & -r_{yu}(0) & r_u(0) \end{bmatrix} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_0 \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} \frac{3r_y(0)-4r_y(1)+r_y(2)}{\Delta^3} \\ \frac{-r_y(0)+2r_y(1)-r_y(2)}{\Delta^2} \\ \frac{r_{yu}(0)-2r_{yu}(1)+r_{yu}(2)}{\Delta^2} \end{bmatrix}. \tag{93}$$

$$\phi_k = \begin{cases} [-\delta y, -y, -y^3, u, uy]^T & \text{(SDRM)} \\ \left[-\delta y, -\left(1+\frac{\Delta}{2}\delta\right) y, -\left(1+\frac{\Delta}{2}\delta\right)(y^3), \right. & \\ \quad \left. \left(1+\frac{\Delta}{2}\delta\right) u, \left(1+\frac{\Delta}{2}\delta\right)(uy)\right]^T & \text{(MIFZD)} \\ [-\delta y, -y, -y^3, u, uy, -\delta(y^3), \delta u, \delta(uy)]^T & \text{(MIPZD)} \end{cases} \tag{85}$$

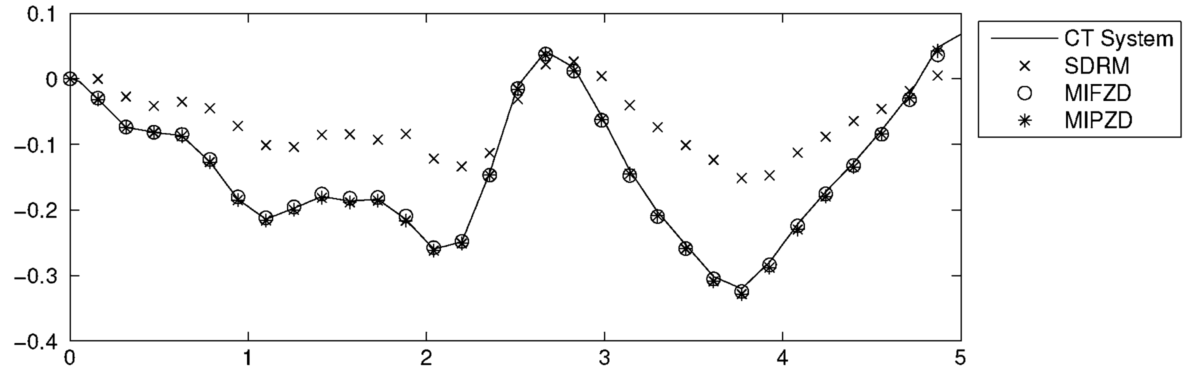

Fig. 1. Simulated output sequences for the validation input.

It can be shown (see [43] for details) that, as the sampling period Δ goes to zero

$$\hat{\alpha}_1 \to \alpha_1 \quad \hat{\alpha}_0 \to \alpha_0 \quad \text{and} \quad \hat{\beta}_0 \to \frac{1}{2}\beta_0. \tag{94}$$

This means that the estimates are clearly asymptotically biased for approximate model (88).

The advantages of including *zero dynamics* in the discrete-time model are further illustrated in the following example.

Example 2: Let us consider the linear system (86), with the following continuous-time parameters $\alpha_1 = 3, \alpha_0 = 2, \beta_0 = 2$.

We performed system identification for the discrete-time model

$$G_\delta(\gamma) = \frac{B_\delta(\gamma)}{\gamma^2 + \hat{\alpha}_1\gamma + \hat{\alpha}_0} \tag{95}$$

where7

$$B_\delta(\gamma) = \begin{cases} \hat{\beta}_0 & (\text{linearSDRM}) \\ \hat{\beta}_0\left(1 + \frac{\Delta}{2}\gamma\right) & (\text{linearMIFZD}) \\ \hat{\beta}_0 + \hat{\beta}_1\gamma & (\text{linearMIPZD}). \end{cases} \tag{96}$$

These models are the linear analogues of the ones used for the nonlinear case in Example 1.

We choose a sampling period $\Delta = \pi/100$ [s] and a random Gaussian input of unit variance. Table III shows the estimation results where the bias is clear in the estimate of β_0 for the SDRM (as predicted in Remark 13). Note that the system considered is linear, thus, the exact discrete-time parameters can be computed for the given sampling period. These are also given in Table III.

Remark 14: The analysis presented in Remark 13 is helpful to understand the presence of asymptotic bias in the SDRM estimates in Examples 1 and 2, for both nonlinear and linear systems. This bias can be mitigated, for example, if we use output error system identification instead of least squares estimation, but at the expense of using nonconvex optimization.

TABLE III
PARAMETER ESTIMATES FOR A LINEAR SYSTEM

	Parameters		Estimates		
	CT	Exact DT	SDRM	MIFZD	MIPZD
α_1	3	2.923	2.8804	2.9471	2.9229
α_0	2	1.908	1.9420	1.9090	1.9083
β_1	–	0.0305	–	$\frac{\beta_0\Delta}{2} = 0.03$	0.0304
β_0	2	1.908	0.9777	1.9090	1.9083

VI. CONCLUSION

This paper has developed an approximate discrete-time model for nonlinear systems. The obtained sampled-data model uses a more sophisticated derivative approximation than the simple Euler approach. Moreover, an insightful interpretation is given in terms of an explicit characterization of the nonlinear *sampling zero dynamics* of the obtained model. This extends a well-known result for sampling zeros of linear systems to the nonlinear case. The result is believed to give important insights which are relevant to many aspects of nonlinear systems theory. By way of illustration, we have shown that models obtained by system identification have higher fidelity when nonlinear *sampling zero dynamics* are included in the model.

APPENDIX I
PROOF OF LEMMA 1

We describe the nth-order integrator $G(s) = s^{-n}$ in state–space form

$$\dot{x} = Ax + Bu \tag{97}$$
$$y = Cx \tag{98}$$

where the matrices take the specific form

$$A = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{99}$$
$$C = [1 \quad 0 \quad \cdots \quad 0]. \tag{100}$$

The equivalent sampled-data model, assuming a ZOH input u_k as in (2), is given by

$$\delta x_k = A_\delta x_k + B_\delta u_k \tag{101}$$
$$y_k = C_\delta x_k \tag{102}$$

where $x_k = x(k\Delta), y_k = y(k\Delta)$, and the matrices can be exactly obtained noting that the matrix A is *nilpotent*, i.e., $A^n = 0$. This yields

$$A_\delta = \frac{e^{A\Delta} - I}{\Delta} = \begin{bmatrix} 0 & 1 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-2}}{(n-1)!} \\ 0 & 0 & 1 & \cdots & \frac{\Delta^{n-3}}{(n-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & 0 & \dots & & 0 \end{bmatrix} \tag{103}$$

$$B_\delta = \frac{1}{\Delta}\int_0^\Delta e^{A\eta} B\, d\eta = \begin{bmatrix} \frac{\Delta^{n-1}}{n!} & \frac{\Delta^{n-2}}{(n-1)!} & \dots & 1 \end{bmatrix}^T \tag{104}$$

$$C_\delta = C = [1 \quad 0 \quad \cdots \quad 0]. \tag{105}$$

Note that, applying the delta transform to (101), with initial conditions equal to zero, we obtain the following set of equations:

$$\begin{bmatrix} \gamma X_1 \\ \gamma X_2 \\ \vdots \\ \gamma X_{n-1} \\ \gamma X_n \end{bmatrix} = \begin{bmatrix} 1 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-2}}{(n-1)!} & \frac{\Delta^{n-1}}{n!} \\ 0 & 1 & \cdots & \frac{\Delta^{n-3}}{(n-2)!} & \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \frac{\Delta}{2!} \\ 0 & & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_n \\ U \end{bmatrix}. \tag{106}$$

This set of algebraic equations can be solved in terms of the first state $X_1(\gamma) = Y(\gamma)$

$$\begin{bmatrix} \gamma Y \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma X_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-1}}{n!} \\ -\gamma & 1 & \cdots & \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -\gamma & 1 \end{bmatrix}}_{M_n} \begin{bmatrix} X_2 \\ \vdots \\ X_n \\ U \end{bmatrix}. \tag{107}$$

Next, using *Cramer's Rule* [41], we can solve the system for the input $U(\gamma)$ in terms of $Y(\gamma)$

$$U = \frac{\det N}{\det M_n} \tag{108}$$

where M_n is defined as in (107) [see also (5)], and

$$N = \begin{bmatrix} 1 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-2}}{(n-1)!} & \gamma Y \\ -\gamma & 1 & \cdots & \frac{\Delta^{n-3}}{(n-2)!} & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & -\gamma & 1 & 0 \\ 0 & \dots & & -\gamma & 0 \end{bmatrix}. \tag{109}$$

From (108), using definition (4), and computing the determinant of the matrix N, for example, along the last column, we obtain the inverse sampled-data system transfer function

$$U(\gamma) = \frac{\gamma^n}{p_n(\Delta\gamma)} Y(\gamma) \Rightarrow G_\delta(\gamma) = \frac{Y(\gamma)}{U(\gamma)} = \frac{p_n(\Delta\gamma)}{\gamma^n}. \tag{110}$$

APPENDIX II
PROOF OF LEMMA 2

We first present the following preliminary result.

Lemma 8: For an integer $n \geq 1$, consider the matrix M_n defined in (5) and (107). Then, we have

$$(M_n)^{-1} \begin{bmatrix} \gamma \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{p_n(\Delta\gamma)} \begin{bmatrix} \gamma p_{n-1}(\Delta\gamma) \\ \gamma^2 p_{n-2}(\Delta\gamma) \\ \vdots \\ \gamma^n \end{bmatrix}. \tag{111}$$

Proof: The left-hand side of (111) corresponds to solving system (107) by inverting the matrix M_n, and omitting the output variable $Y(\gamma)$. Thus, in the same way that we solved (107) for $U(\gamma)$ in Appendix I, we can use Cramer's Rule [41] to solve for every state $X_\ell, \ell = 2, \dots, n$. This leads to

$$X_\ell = \frac{\det N_{\ell-1}}{\det M_n} Y, \qquad \ell = 2, \dots, n \tag{112}$$

where $N_{\ell-1}$ is the matrix obtained by replacing the $(\ell-1)$th column of M_n by the vector on the left of (107). Thus, (113), as shown at the bottom of the next page, holds.

Then, computing the determinant along the $(\ell-1)$th column, we have that

$$\det N_{\ell-1} = \gamma(-1)^\ell (\det P)(\det M_{n-\ell+1}) \tag{114}$$

where

$$P = \begin{bmatrix} -\gamma & 1 & \dots & \frac{\Delta^{\ell-4}}{(\ell-3)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & -\gamma & 1 \\ 0 & \cdots & 0 & -\gamma \end{bmatrix} \Rightarrow \det P = (-\gamma)^{\ell-2} \tag{115}$$

and, from definition (4)–(5)

$$\det M_{n-\ell+1} = p_{n-\ell+1}(\Delta\gamma). \tag{116}$$

Replacing (115) and (116) in (114), we obtain

$$\det N_{\ell-1} = \gamma^{\ell-1} p_{n-\ell+1}(\Delta\gamma). \tag{117}$$

It follows that the solution of (107) is

$$\begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_n \\ U \end{bmatrix} = (M_n)^{-1} \begin{bmatrix} \gamma \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} Y$$

$$= \frac{1}{p_n(\Delta\gamma)} \begin{bmatrix} \gamma p_{n-1}(\Delta\gamma) \\ \gamma^2 p_{n-2}(\Delta\gamma) \\ \vdots \\ \gamma^{n-1} p_1(\Delta\gamma) \\ \gamma^n \end{bmatrix} Y \tag{118}$$

which is equivalent to (111). □

We now proceed to establish Lemma 2. From the definition of the matrix M_n in (5), we have that

$$M_n = \left[\begin{array}{c|ccc} 1 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-1}}{n!} \\ \hline -\gamma & & & \\ \vdots & & M_{n-1} & \\ 0 & & & \end{array}\right]. \tag{119}$$

The determinant of this matrix can be readily computed, using the *matrix inversion lemma* (see, for example, [42, App. E])

$$\det M_n = \det M_{n-1} \times \det\left(1 - \begin{bmatrix} \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-1}}{n!} \end{bmatrix} (M_{n-1})^{-1} \begin{bmatrix} -\gamma \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right). \tag{120}$$

Finally, from (4) and using Lemma 8, we have that

$$p_n = p_{n-1} \times \left(1 + \begin{bmatrix} \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-1}}{n!} \end{bmatrix} \frac{1}{p_{n-1}} \begin{bmatrix} \gamma p_{n-2} \\ \vdots \\ \gamma^{n-2} p_1 \\ \gamma^{n-1} \end{bmatrix}\right) \tag{121}$$

where we have replaced $p_\ell(\Delta\gamma)$ by p_ℓ. The recursive relation in (7) corresponds exactly to (121).

Equation (8) readily follows from recursion (7), noting that

$$\lim_{\Delta\to 0} p_n(\Delta\gamma) = \lim_{\Delta\to 0} p_{n-1}(\Delta\gamma) = \cdots = \lim_{\Delta\to 0} p_1(\Delta\gamma) = 1. \tag{122}$$

Appendix III
Proof of Lemma 6

(i)–$(ii) \Rightarrow (a)$–(b)

Using the delta operator definition (1), we have that

$$\begin{aligned} \frac{\partial(\delta^\ell y_k)}{\partial u_k} &= \frac{\partial}{\partial u_k}\left(\left(\frac{q-1}{\Delta}\right)^\ell y_k\right) \\ &= \frac{1}{\Delta^\ell}\frac{\partial}{\partial u_k}\sum_{i=0}^{\ell}\binom{\ell}{i}(-1)^{\ell-i}(q^i y_k) \\ &= \frac{1}{\Delta^\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}(-1)^{\ell-i}\frac{\partial y_{k+i}}{\partial u_k} \end{aligned} \tag{123}$$

where clearly if $(\partial y_{k+i}/\partial u_k) = 0$, for all $i = 0, \ldots, r-1$, then $(\partial(\delta^\ell y_k))/(\partial u_k) = 0$, for all $\ell = 0, \ldots, r-1$. Furthermore, $(\partial y_{k+r}/\partial u_k) \neq 0$ implies $(\partial(\delta^r y_k))/(\partial u_k) \neq 0$.

(a)–$(b) \Rightarrow (i)$–(ii)

This follows from similar arguments, on noting that $q = 1 + \Delta\delta$. Then, we have that

$$\begin{aligned} \frac{\partial y_{k+\ell}}{\partial u_k} = \frac{\partial(q^\ell y_k)}{\partial u_k} &= \frac{\partial}{\partial u_k}((1+\Delta\delta)^\ell y_k) \\ &= \frac{\partial}{\partial u_k}\left(\sum_{i=0}^{\ell}\binom{\ell}{i}\Delta^i\delta^i y_k\right) \\ &= \sum_{i=0}^{\ell}\binom{\ell}{i}\Delta^i\frac{\partial(\delta^i y_k)}{\partial u_k} \end{aligned} \tag{124}$$

where clearly if $(\partial(\delta^i y_k))/(\partial u_k) = 0$, for all $i = 0, \ldots, r-1$, then $(\partial y_{k+\ell}/\partial u_k) = 0$ for all $\ell = 0, \ldots, r-1$. Furthermore, $(\partial(\delta^r y_k))/(\partial u_k) \neq 0$ implies $(\partial y_{k+r}/\partial u_k) \neq 0$.

Appendix IV
Proof of Lemma 7

We consider the sampled-data model for the nth order integrator given by (101)–(105), and the state–space similarity

$$N_{\ell-1} = \left[\begin{array}{ccc|c|ccc} 1 & \cdots & \frac{\Delta^{\ell-3}}{(\ell-2)!} & \gamma & \frac{\Delta^{\ell-1}}{\ell!} & \cdots & \frac{\Delta^{n-1}}{n!} \\ \hline -\gamma & \ddots & & 0 & \frac{\Delta^{\ell-2}}{(\ell-1)!} & \cdots & \frac{\Delta^{n-2}}{(n-1)!} \\ 0 & \ddots & 1 & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & -\gamma & 0 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-\ell+1}}{(n-\ell+2)!} \\ \hline 0 & \cdots & 0 & 0 & 1 & \cdots & \frac{\Delta^{n-\ell}}{(n-\ell+1)!} \\ \vdots & \ddots & \vdots & 0 & -\gamma & \ddots & \frac{\Delta^{n-\ell-1}}{(n-\ell)!} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right] \tag{113}$$

transformation $z = Tx$, where the nonsingular matrix T is given by

$$T = \left[\begin{array}{c|c} 1 & 0 \\ \hline T_{21} & I_{n-1} \end{array}\right] \iff T^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline -T_{21} & I_{n-1} \end{array}\right] \tag{125}$$

where

$$T_{21} = \left[-\frac{n}{\Delta} \quad \cdots \quad -\frac{n!}{\Delta^{n-1}}\right]^T. \tag{126}$$

Then, the new state–space representation is given by the following matrices:

$$\begin{aligned} \bar{A}_\delta = TA_\delta T^{-1} &= \left[\begin{array}{c|c} q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} -A_{12}T_{21} & A_{12} \\ \hline -(T_{21}A_{12} + A_{22})T_{21} & T_{21}A_{12} + A_{22} \end{array}\right] \end{aligned} \tag{127}$$

where, from (103)

$$A_{12} = \left[1 \quad \frac{\Delta}{2} \quad \cdots \quad \frac{\Delta^{n-2}}{(n-1)!}\right] \tag{128}$$

$$A_{22} = \begin{bmatrix} 0 & 1 & \cdots & \frac{\Delta^{n-3}}{(n-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{129}$$

and

$$\bar{B}_\delta = TB_\delta = \left[\frac{\Delta^{n-1}}{n!} \quad 0 \quad \cdots \quad 0\right]^T \tag{130}$$

$$\bar{C}_\delta = C_\delta T^{-1} = C_\delta. \tag{131}$$

These state–space matrices give the *normal form* that appears in (39)–(40).

To prove (41), we first note that

$$M_n \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} \frac{\Delta^{n-1}}{n!} & A_{12} \\ \hline -\frac{\Delta^{n-1}}{n!}T_{21} & A_{22} - \gamma I_{n-1} \end{array}\right]. \tag{132}$$

Computing the determinant of the matrices on both sides of the equation and using the *matrix inversion lemma* (see, for example, [42, App. E]), we have that

$$(\det M_n)(-1)^{n-1} = \frac{\Delta^{n-1}}{n!}\det(A_{22} - \gamma I_{n-1} + T_{21}A_{12}) \tag{133}$$

where, from definition of Q_{22} in (127), we finally have that

$$\begin{aligned} \det M_n &= \frac{\Delta^{n-1}}{n!}(-1)^{n-1}\det(-\gamma I_{n-1} + (A_{22} + T_{21}A_{12})) \\ &= \frac{\Delta^{n-1}}{n!}\det(\gamma I_{n-1} - Q_{22}). \end{aligned} \tag{134}$$

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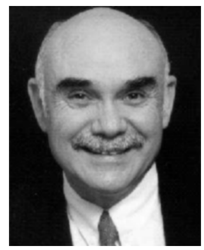
Juan I. Yuz (S'03) received the professional title of Ingeniero Civil Electrónico and the M.S. degree in electronics engineering from Universidad Técnica Federico Santa María, Valparaíso, Chile, in 2001, obtaining the Best Electronics Engineering Student Award. Since 2002, he has been working toward the Ph.D. degree in electrical engineering at The University of Newcastle, NSW, Australia.

His research areas include sampled-data systems, system identification, constrained control, and performance limitations.



Graham C. Goodwin (M'74–SM'84–F'86) received the B.Sc (physics), B.E. (electrical engineering), and Ph.D. degrees from the University of New South Wales, Australia, in 1965, 1967, and 1971, respectively.

From 1970 to 1974, he was a Lecturer in the Department of Computing and Control, Imperial College, London, U.K. Since 1974, he has been with the School of Electrical Engineering and Computer Science, The University of Newcastle, Australia. He is currently Professor of Electrical Engineering and Research Director of the Centre of Complex Dynamics Systems and Control (CDSC). He is the coauthor of eight monographs, four edited volumes, and several hundred technical papers.

Dr. Goodwin is the recipient of several international prizes including the IEEE Control Systems Society 1999 Hendrik Bode Lecture Prize, a Best Paper award by the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, a Best Paper award by the *Asian Journal of Control*, and Best Engineering Text Book award from the International Federation of Automatic Control. He was also the recipient of an ARC Federation Fellowship, and is an Honorary Fellow of Institute of Engineers, Australia, a Fellow of the Australian Academy of Science, a Fellow of the Australian Academy of Technology, Science and Engineering, a Member of the International Statistical Institute, a Fellow of the Royal Society, London, U.K., and a foreign member of the Royal Swedish Academy of Sciences.