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The Effect of Regularization on Variance Error

Brett Ninness and Håkan Hjalmarsson

***Abstract*—This note addresses the problem of quantifying the effect of noise induced error(so called "variance error") in system estimates found via a regularised cost criterion. It builds on recent work by the authors in which expressions for nonregularised criterions are derived which are exact for finite model order. Those new expressions were established to be very different to previous quantifications that are widely used but based on asymptotic in model order arguments. A key purpose of this note is to expose a rapprochement between these new finite model order, and the preexisting asymptotic model order quantifications. In so doing, a further new result is established. Namely, that variance error in the frequency domain is dependent on the choice of the point about which regularization is affected.**

***Index Terms*—Orthonormal bases, parameter estimation, system identification, variance error.**

I. Introduction

When performing system identification via the widely used prediction-error method with a quadratic criterion [1], [2], then a seminal result is that under open-loop conditions the noise-induced error, as measured by the variability of the ensuing frequency response estimate $G\left(e^{j\omega}, \widehat{\theta}_N^n\right)$, may be quantified via the following approximation [1], [3]–[5]:

$$\mathrm{Var}\left\{G\left(e^{j\omega}, \widehat{\theta}_N^n\right)\right\} \approx \frac{m}{N}\frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}. \tag{1}$$

Here, Φ_ν and Φ_u are, respectively, the measurement noise and input excitation power spectral densities, and $\widehat{\theta}_N^n$ is the prediction error estimate based on N observed data points of a vector $\theta^n \in \mathbf{R}^n$ that parameterizes a model structure $G(q, \theta^n)$ for which(essentially) the model order $m = \dim \theta^n/(2^d)$ where d is the number of denominator polynomials to be estimated in the model structure.

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A fundamental aspect of the approximation (1) is that it is derived by taking the limiting value of the variance as model order m tends to infinity, and then employing that limiting value as an approximation for finite m.

Motivated by the desire to improve the accuracy of variance error quantifications, [6] and [7] have derived new expressions that are exact for finite model order (although they are still based on limiting arguments with respect to observed data length N).

As discussed in [6], there can be very large discrepancies between the new quantifications derived for finite-model order [6], and the approximation (1); [6] illustrates orders of magnitude difference on a simple example.

A key purpose of this note is to address this issue and provide a rapprochement between the results. The approach taken here is to derive new quantifications that are exact for finite model order. Although finite, this order may also be arbitrarily large, provided an appropriate regularised criterion is used to ensure that at the arbitrarily large model order, the limiting (in N) estimate is uniquely defined.

Essentially, via this strategy, the work here establishes that when the regularising point (in parameter space) implies that any pole zero cancellations in the estimated model are constrained to be at the origin, then as model order m increases, the "exact"(for finite-model order) variance expression becomes arbitrarily close to the well known approximation (1). However, when the pole zero cancellations are not at the origin, the rapprochement is lost. This fact exposes the further new result that variance error (in the frequency domain) is dependent on the point about which regularization is imposed.

As overview of the organization of this note, Section II makes concrete the estimation algorithms and model structures being considered. Certain key ideas, notation and definitions are also introduced. Section III presents the main technical results, which are new variance error quantifications that are novel in that they do not depend on asymptotic in model order arguments, yet they still apply for model orders possibly greater than that of an underlying true system. Section IV discusses the ramifications and practical consequences of these results and, in particular, uses them to argue a rapprochement between new finite model order expressions [6] and pre-existing asymptotic model order approximations [3]. Section V provides concluding remarks and comments about prospective future studies.

II. Problem Formulation

In what follows, it is assumed that the relationship between an observed input data record $\{u_t\}$ and output data record $\{y_t\}$ obeys

$$\mathcal{S} : y_t = G(q)u_t + \nu_t \quad \nu_t = H(q)e_t \tag{2}$$

and that this is modeled according to

$$\mathcal{M} : y_t = G(q, \theta^n)u_t + H(q, \theta^n)e_t \tag{3}$$

where the "dynamics model" $G(q, \theta^n)$ and the "noise model" $H(q, \theta^n)$ are jointly parametrized by a vector $\theta^n \in \mathbf{R}^n$ and are of the rational forms ($A(q, \theta^n) - D(q, \theta^n)$ that follow are all polynomials in the backward shift operator q^{-1})

$$G(q, \theta^n) = \frac{B(q, \theta^n)}{A(q, \theta^n)} \quad H(q, \theta^n) = \frac{C(q, \theta^n)}{D(q, \theta^n)} \tag{4}$$

while $\{e_t\}$ in (3) is a zero-mean white noise sequence that satisfies $\mathbf{E}\left\{e_t^2\right\} = \sigma^2, \mathbf{E}\left\{|e_t|^8\right\} < \infty$.

The postulated relationship (3) can encompass a range of model structures such as FIR, ARMAX, "Output-Error," and "Box-Jenkins" [1], [2], [8]. For all these cases, since $H(q, \theta^n)$ is also constrained to be monic (i.e., $\lim_{|q|\to\infty} H(q,\theta^n) = 1$) for all θ, then the mean-square optimal one-step ahead predictor $\widehat{y}_t(\theta^n)$ based on the model structure (3) is [1]

$$\widehat{y}_t(\theta^n) = H^{-1}(q,\theta^n)G(q,\theta^n)u_t + \left[1 - H^{-1}(q,\theta^n)\right] y_t \tag{5}$$

with associated prediction error

$$\varepsilon_t(\theta^n) \triangleq y_t - \widehat{y}_t(\theta^n) = H^{-1}(q,\theta^n)\left[y_t - G(q,\theta_n)u_t\right]. \tag{6}$$

Using this, a regularised quadratic estimation criterion may be defined as

$$V_N(\theta^n,\delta) = \frac{1}{2N}\sum_{t=1}^{N}\varepsilon_t^2(\theta^n) + \frac{\delta}{2}\|\theta^n - \theta_\circ^n\|^2 \tag{7}$$

and then used to construct the prediction error estimate $\widehat{\theta}_N^n(\delta)$ of θ^n as

$$\widehat{\theta}_N^n(\delta) \triangleq \arg\min_{\theta^n\in\Theta} V_N(\theta^n,\delta) \tag{8}$$

where $\Theta \subset \mathbf{R}^n$ is compact. In (7) the norm $\|\cdot\|$ is the Euclidean one, and $\delta > 0$ is a so-called "regularising" parameter.

The point of using the regularised criterion is that in the situation considered in this note of the model structure $\mathcal{M}$ being of the Box-Jenkins or Output-Error form then, as established in [1], [9] under mild assumptions, the unregularised ($\delta = 0$) estimate $\widehat{\theta}_N^n(0)$ converges with increasing N according to

$$\lim_{N\to\infty}\widehat{\theta}_N^n(0) = \theta_\circ^n \triangleq \arg\min_{\theta^n\in\Theta}\lim_{N\to\infty}\mathbf{E}\left\{V_N(\theta^n,0)\right\} \quad \text{w.p.1.} \tag{9}$$

Note that this limiting estimate $\theta_\circ^n$ is not uniquely defined in cases where the true model order for the system $\mathcal{S}$ is less than that of the model structure $\mathcal{M}$ since pole-zero pairs beyond those necessary to capture the underlying true dynamics may cancel in an infinite variety of ways while still delivering the same transfer function.

At the same time, the purpose of this note is to study the relationship between the quantification (1) and the asymptotic in model order result which generates it; viz. [1], [3]–[5]

$$\lim_{m\to\infty}\lim_{N\to\infty}\frac{N}{m}\mathrm{Var}\left\{G\left(e^{j\omega},\widehat{\theta}_N^n\right)\right\} = \frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}. \tag{10}$$

Hence, consideration of the overmodeled situation is unavoidable. In order to address it, while also ensuring that the limiting estimate $\theta_\circ^n$ is well defined, the regularised criterion (7) is employed, with $\theta_\circ^n$ being any value such that $\varepsilon_t(\theta_\circ^n) = e_t$.

One contribution of this note is to establish that, while there is freedom in the selection of the regularization point $\theta_\circ^n$, the precise choice of $\theta_\circ^n$ (in terms of the excess pole-zero cancellations it implies) can have a significant effect on estimate variability in the frequency domain.

At this point it is worth noting that the regularization approach taken here is the identical to that employed in the original work [3]. There it was emphasised that there is no practical difficulty implied by the fact that $\widehat{\theta}_N^n(\delta)$ defined as the minimizer of (7) is unrealisable since $\theta_\circ^n$ is unknown to the user. It is introduced here (and in [3]) merely to provide a technical artifice for defining the unique estimate $\lim_{\delta\to 0} G\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right)$.

With this in mind, this note employs the fact that as N increases, the estimate $\widehat{\theta}_N^n$ converges in law to a normally distributed random variable with mean value $\theta_\circ^n$ according to [1], [8], and [10]

$$\sqrt{N}\left(\widehat{\theta}_N^n - \theta_\circ^n\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,P_n) \text{ as } N\to\infty \tag{11}$$

and, furthermore, under the added assumption of $\mathbf{E}\left\{|e_t|^8\right\} < \infty$ then as established in [1, App. 9B]

$$\lim_{N\to\infty}\mathrm{Var}\left\{\widehat{\theta}_N^n - \theta_\circ^n\right\} = P_n \tag{12}$$

where, in the particular case considered in this note of the model structure (3) being rich enough to encompass any true underlying dynamics [1]

$$P_n^{-1} = \frac{1}{\sigma^2}\mathbf{E}\left\{\psi_t\left(\theta_\circ^n\right)\psi_t^T\left(\theta_\circ^n\right)\right\}. \tag{13}$$

Furthermore, for some matrix of transfer functions $\Pi(q,\theta^n)$, and some quasistationary (possibly vector valued) signal $\zeta_t(\theta^n)$

$$\begin{aligned}\psi_t(\theta^n) &\triangleq -\frac{d}{d\theta^n}\widehat{y}_t(\theta^n)\\ &= -H^{-1}(q,\theta^n)\frac{d\Pi(q,\theta^n)}{d\theta^n}\zeta_t(\theta^n).\end{aligned} \tag{14}$$

Unfortunately, while this explicit formulation of P_n exists, in general it does not provide significant insight into how various design variables affect the accuracy of the estimated frequency functions $G\left(e^{j\omega},\widehat{\theta}_N^n\right)$ and $H\left(e^{j\omega},\widehat{\theta}_N^n\right)$. In response to this, [1], [3]–[5], and [11] have used an approach of investigating how (11) manifests itself in the variability $\Delta_n(\omega)$ of $[G\left(e^{j\omega},\widehat{\theta}_N^n\right), H\left(e^{j\omega},\widehat{\theta}_N^n\right)]^T$; the result being approximations such as (1).

Central to the contribution of [6] is the novel approach of recognising that the problem of quantifying $\Delta_n(\omega)$ is closely related to the problem of quantifying the reproducing kernel for a certain space X_n which is defined via the rows of the matrix (θ^n is assumed to be a column vector)

$$\Psi\left(z,\theta_\circ^n\right) \triangleq H^{-1}\left(z,\theta_\circ^n\right)\left.\frac{d\Pi(z,\theta^n)}{d\theta^n}\right|_{\theta^n=\theta_\circ^n} S_{\zeta_\circ}(z) \tag{15}$$

according to

$$X_n \triangleq \mathrm{Span}\left\{\left[\Psi\left(z,\theta_\circ^n\right)\right]_1^T,\ldots,\left[\Psi\left(z,\theta_\circ^n\right)\right]_n^T\right\} \tag{16}$$

and where, in (15), the term $S_{\zeta_\circ}(z)$ is a spectral factor (under mild assumptions, this factor will be unique) associated with the process $\{\zeta_t\left(\theta_\circ^n\right)\}$.

The space X_n for certain important model structures such as Box-Jenkins, Output-Error, and ARMAX was derived in [6], to which we refer the reader for more detail.

In relation to this, the fundamental quantity associated with this space X_n of $\mathbf{C}^p$ valued functions termed the "reproducing kernel" $\varphi_n(\lambda,\omega):[-\pi,\pi]\times[-\pi,\pi]\to\mathbf{C}^{p\times p}$ is an entity such that for any $\alpha\in\mathbf{C}^p$ [6]

$$\varphi_n(\cdot,\omega)\alpha \in X_n \qquad \forall\,\omega\in[-\pi,\pi] \tag{17}$$

and for any $f\in X_n$

$$\langle f(\cdot),\varphi_n(\cdot,\omega)\alpha\rangle = \alpha^\star f(\omega) \tag{18}$$

where the previous inner product is defined for arbitrary functions $f,g:[-\pi,\pi]\to\mathbf{C}^p$ according to

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} g^\star(\lambda)f(\lambda)d\lambda. \tag{19}$$

While it is essential to introduce the reproducing kernel $\varphi(\lambda,\omega)$ here, since it will be fundamental to the quantification of the variance error of regularised estimates, the reader is referred to the companion work [6] for a more complete discussion of it together with concrete expressions.

III. Main Results

The main result of this note is a quantification of frequency domain variability $\Delta_n(\omega)$ that is not asymptotic in the model order m. In fact, the following theorem establishes that the fundamental variance quantification provided in [6, Th. 5.1] applies also for the situation considered in this note of overmodeling combined with a regularised estimation criterion.

Theorem 3.1: Suppose that $\widehat{\theta}_N^n$ is calculated via the regularised criterion (7) and using the model structure (3). Suppose further that the following assumptions are satisfied.

1) $\varepsilon_t(\theta_\circ^n) = e_t$ where $\{e_t\}$ is a zero mean i.i.d. process that satisfies $\mathbf{E}\{|e_t|^8\} < \infty$.
2) The relationship (14) holds for some $\Pi(q,\theta^n)$, and some quasistationary (possibly vector valued) signal $\{\zeta_t(\theta^n)\}$ and for which the power spectral density $\Phi_{\zeta_\circ}(\omega)$ of $\{\zeta_t(\theta_\circ^n)\}$ satisfies $\Phi_{\zeta_\circ}(\omega) > 0 \; \forall\, \omega \in [-\pi,\pi]$.

Then

$$\lim_{\delta\to 0}\lim_{N\to\infty} N\cdot \mathrm{Cov}\left\{\begin{bmatrix} G\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right) \\ H\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right)\end{bmatrix}\right\} = \Delta_n(\omega) \tag{20}$$

where

$$\Delta_n(\omega) = \Phi_\nu(\omega) S_{\zeta_\circ}^{-\star}(e^{j\omega})\varphi_n(\omega,\omega)S_{\zeta_\circ}^{-1}(e^{j\omega}) \tag{21}$$

with $\varphi_n(\lambda,\omega)$ being the reproducing kernel for the space X_n defined via (15) and (16). □

Proof: See Appendix I. □

While this core result applies for any model structure that can be cast in the form (3), for the sake of concreteness it is worthwhile to consider certain specific cases as a corollary to this main result. For this purpose, the model structure (3) parametrized with polynomials of the form

$$A(q,\theta^n) = 1 + a_1 q^{-1} + a_2 q^{-1} + \cdots + a_{m_a} q^{-m_a} \tag{22}$$
$$B(q,\theta^n) = b_1 q^{-1} + b_2 q^{-1} + \cdots + b_{m_b} q^{-m_b} \tag{23}$$
$$D(q,\theta^n) = 1 + d_1 q^{-1} + d_2 q^{-1} + \cdots + d_{m_d} q^{-m_d} \tag{24}$$
$$C(q,\theta^n) = 1 + c_1 q^{-1} + c_2 q^{-1} + \cdots + c_{m_c} q^{-m_c} \tag{25}$$

will be assumed when a Box-Jenkins model structure is referred to, while the model structure (3) parametrized with the numerator and denominator polynomials of the form (22) and (23) and

$$C(q,\theta^n) = D(q,\theta^n) = 1 \tag{26}$$

will be assumed when we refer to an Output-Error model structure.

Corollary 3.1: Suppose that the conditions of Theorem 3.1 are satisfied together with the following further assumptions.

1) e_t and u_t are jointly quasistationary with cross-spectrum $\Phi_{ue} \equiv 0$.
2) $G(z,\theta_\circ^n)$ and $H(z,\theta_\circ^n)$ may be written as

$$G(z,\theta_\circ^n) = \frac{B(z)}{A(z)}\cdot\frac{T_G(z)}{T_G(z)} \quad H(z,\theta_\circ^n) = \frac{C(z)}{D(z)}\cdot\frac{T_H(z)}{T_H(z)} \tag{27}$$

where all terms to the right of the equals signs in (27) are polynomials in z^{-1} with $B(z)$, $A(z)$ being relatively prime and $C(z)$, $D(z)$ being relatively prime.
3) $T_G(z)$ and $T_H(z)$ are of orders m_{t_g} and m_{t_h}, respectively, and, as previously, m_a, m_d, m_b, m_c denote (respectively) the denominator and numerator orders of $G(z,\theta^n)$ and $H(z,\theta^n)$.
4) With $F(z)$ being a spectral factor of $\Phi_u(\omega)$, then $A_\dagger(z)$ defined as

$$A_\dagger(z) = A^2(z)T_G(z)\frac{C(z)}{D(z)F(z)} \tag{28}$$

is a polynomial in z^{-1} of degree no greater than $m_a + m_b - m_{t_g}$.

Then, with $\{\xi_1,\ldots,\xi_{m_a+m_b-m_{t_g}}\}$ being the zeros of $z^{m_a+m_b-m_{t_g}}A_\dagger(z)$ and $\{\eta_1,\ldots,\eta_{m_c+m_d-m_{t_h}}\}$ being the zeros of $z^{m_c+m_d-m_{t_h}}D(z)C(z)T_H(z)$, if a Box-Jenkins model structure is employed, then

$$\lim_{\delta\to 0}\left[\lim_{N\to\infty} N\cdot Cov\left\{\begin{bmatrix} G\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right) \\ H\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right)\end{bmatrix}\right\}\right] = \Phi_\nu(\omega)\begin{bmatrix} \frac{\kappa(\omega)}{\Phi_u(\omega)} & 0 \\ 0 & \frac{\widetilde{\kappa}(\omega)}{\sigma^2}\end{bmatrix} \tag{29}$$

where $\kappa(\omega)$ and $\widetilde{\kappa}(\omega)$ are defined by the $\{\xi_k\}$ and $\{\eta_k\}$ according to

$$\kappa(\omega) \triangleq \sum_{k=1}^{m_a+m_b-m_{t_g}} \frac{1-|\xi_k|^2}{|e^{j\omega}-\xi_k|^2}$$
$$\widetilde{\kappa}(\omega) \triangleq \sum_{k=1}^{m_c+m_d-m_{t_h}} \frac{1-|\eta_k|^2}{|e^{j\omega}-\eta_k|^2}. \tag{30}$$

Alternatively, if an Output Error model structure is employed, and the condition $H(z,\theta_\circ^n) = 1$ substituted into (28) results in $A_\dagger(z)$ being a polynomial of order no greater that $m_a + m_b - m_{t_g}$, then with $\kappa(\omega)$ begin defined as just described in the Box-Jenkins case, it holds that

$$\lim_{\delta\to 0}\lim_{N\to\infty} N\cdot \mathrm{Var}\left\{G\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right)\right\} = \frac{\sigma^2}{\Phi_u(\omega)}\cdot\kappa(\omega) \tag{31}$$

for the Output-Error case.

Proof: See Appendix II. □

IV. Discussion

To explore some the implications of this result, let us consider the situation addressed in the original result [3] wherein $m_b = m_a = m$. Also, suppose that $\Phi_u(\omega)$ is white so that $F(z) = \mu$ a constant, and that for the moment we restrict attention to the case of Output-Error systems in which $H(q) = H(q,\theta_\circ^n) = 1$.

In this case, (28) becomes $A_\dagger(z) = A^2(z)T_G(z)$ which is a polynomial in z^{-1} of order $2m_a - m_{t_g}$, and which under the assumption $m_b = m_a$ is of order equal to $m_a + m_b - m_{t_g}$. Therefore, all the assumptions of Corollary 3.1 are satisfied, so that denoting the true underlying system order as $m_\star = m_a - m_{t_g}$ and defining the associated factorizations

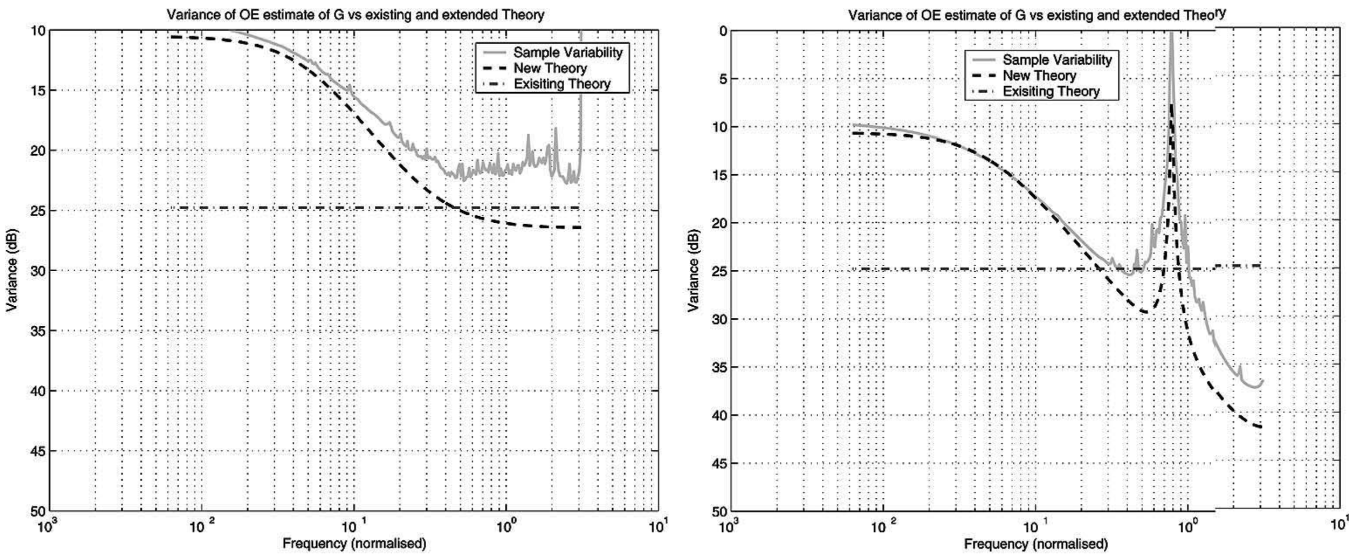
$$A(z) = \prod_{k=1}^{m_\star}\left(1-\xi_k z^{-1}\right) \quad T_G(z) = \prod_{k=1}^{m-m_\star}\left(1-\zeta_k z^{-1}\right) \tag{32}$$

leads to a variance approximation (exact with respect to the finite-model order m) given by Corollary 3.1 as

$$\mathrm{Var}\left\{G\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right)\right\} \approx \frac{1}{N}\frac{\sigma^2}{\Phi_u(\omega)}\times \left[2\cdot\sum_{k=1}^{m_\star}\frac{1-|\xi_k|^2}{|e^{j\omega}-\xi_k|^2} + \sum_{k=1}^{m-m_\star}\frac{1-|\zeta_k|^2}{|e^{j\omega}-\zeta_k|^2}\right] \tag{33}$$

where the factor of 2 arises in (33) since $A_\dagger(z) = A^2(z)T_G(z)$ and hence the zeros $\{\xi_k\}$ of $A(z)$ appear twice in the zeros of $A_\dagger(z) = A^2(z)$.

The essential point now is that even though, by virtue of the pole-zero cancellations in the common set $T_G(z)$, the transfer function $\lim_{\delta\to 0} G\left(e^{j\omega},\widehat{\theta}_N^n(\delta)\right)$ is uniquely defined, the same is *not* true for the common zeros in $T_G(z)$. They are not unique, and depend only on the choice of the regularization point $\theta_\circ^n$, which is constrained only in that $\varepsilon_t(\theta_\circ^n) = e_t$.



(a) (b)

Fig. 1. Figures illustrating variability of regularised output-error estimates. The solid line is the Monte Carlo estimate of the true variability, the dash-dot line is the pre-existing approximation (1) which does not account for system poles or model structure. The dashed line is the new approximation presented in (33) whereby estimated system pole positions $\{\xi_0, \ldots, \xi_{m-1}\}$ and the excess pole-zero cancellations ζ_k inherent in the regularization are accounted for. (a) $\zeta_k = 0$. (b $\zeta_k = 0.99e^{j\pm\pi/4}$.

If the zeros of $T_G(z)$ are taken all at the origin, then the associated choice $\{\zeta_k\} = 0$ in (33) implies that for any finite-model order m

$$\operatorname{Var}\left\{G\left(e^{j\omega}, \widehat{\theta}_N^n(\delta)\right)\right\} \approx \frac{1}{N}\frac{\sigma^2}{\Phi_u(\omega)}\left[2 \cdot \sum_{k=1}^{m_\star} \frac{1-|\xi_k|^2}{|e^{j\omega}-\xi_k|^2} + (m - m_\star)\right]. \quad (34)$$

Now, if $m \approx m_\star$ then the first, frequency dependent term in the square brackets will dominate the variance quantification. On the other hand, if $m \gg m_\star$ then the second term will dominate and the variance quantification will become

$$\operatorname{Var}\left\{G\left(e^{j\omega}, \widehat{\theta}_N^n(\delta)\right)\right\} \approx \frac{m}{N}\frac{\sigma^2}{\Phi_u(\omega)} \quad (35)$$

which is the one originally derived in [3] via an approach of allowing $m \to \infty$ and the same regularization approach as used in this note.

However, an important dividend of Theorem 3.1 is that, via the quantification (33) it provides important insight into the nature of this latter approximation (35), (1). Namely, that (35), (1) is largely determined by the second term in (33), which has no relation to the underlying estimation problem since it is not uniquely defined by it. Instead, it is purely a function of the regularization.

For example, one could just as easily take $\zeta_k \neq 0$, and then the same asymptotic in m argument would yield an approximation

$$\operatorname{Var}\left\{G\left(e^{j\omega}, \widehat{\theta}_N^n(\delta)\right)\right\} \approx \frac{1}{N}\frac{\sigma^2}{\Phi_u(\omega)} \sum_{k=1}^{m-m_\star} \frac{1-|\zeta_k|^2}{|e^{j\omega}-\zeta_k|^2} \quad (36)$$

which is arbitrary, since the $\{\zeta_k\}$ are arbitrary.

This is illustrated in Fig. 1, where an Output-Error model of order $m_a = m_b = 3$ is fitted to data generated by the first-order system

$$G(q) = \frac{0.05}{q - 0.95} \quad (37)$$

on the basis of observing an $N = 10\,000$ sample input-output data record where the output $\{y_t\}$ is corrupted by white Gaussian noise of variance $\sigma^2 = 10$, and where the input $\{u_t\}$ is a realization of a stationary, zero mean, unit variance white Gaussian process.

Since the system is overmodeled, a regularised estimate (7) is employed with regularization parameter $\delta = 10^{-14}$.

When the regularization point $\theta_\circ^n$ is chosen so that any excess pole zero cancellations are at $\zeta_k = 0$, then the true variability (again obtained by Monte Carlo average over 1000 experiment realizations) together with the existing quantification (1) and the new quantification (33) is shown in Fig. 1(a). The y-axis range has been specifically chosen to match that of [6, Fig. 1(a)] in order to aid a comparison by the reader which illustrates that the effect of the regularization zeros at the origin is to reduce the discrepancy between the true variability and the approximation (1).

However, if the regularization zeros are chosen at $\zeta_1 = 0.99e^{j\pi/4}$, $\zeta_2 = 0.99e^{-j\pi/4}$ then the variability and quantifications are shown in Fig. 1(b) to be such that the quantification (1) suffers from orders of magnitude inaccuracy at low, mid, and high frequencies. This is despite the fact that the model order m is triple the underlying true one and, hence, might be considered "large enough" for approximate convergence in (10) and, hence, accuracy of (1).

Clearly, in both cases the underlying true system and experimental conditions are the same and, hence, the discrepancy between Fig. 1(a) and (b) shows that, despite what might be expected, the quantification (1) does not wholly elicit features inherent to the estimation problem. Instead it may exhibit (possibly dominant) features that are entirely nonintrinsic to the estimation problem, namely, the regularization point.

Although the preceding argument was developed under special assumptions on $\Phi_u(\omega)$, the note concludes that in the case of Output-Error or Box-Jenkins modeling, there is strong evidence that the variance quantification (1) is one that is generically dominated by the choice of a particular regularising point. This implies that when using these model structures, it could be inappropriate to employ the approximation (35), (1) in situations where no regularization has been introduced or, if it has, the model order is not significantly higher than what is believed to be the underlying true one.

V. Conclusion

This note was motivated by a desire to reconcile pre-existing variance expressions, based on asymptotic in model order argument, with new variance quantifications that are "exact" for finite model order.

Since a setting of either Box-Jenkins or Output-Error structures was of interest, this necessitated the use of regularization. Hence, the main theoretical results here can be viewed as contributions to a study of how regularization affects estimation accuracy as judged by variability in the frequency domain.

In this context, a key new finding was that the choice of regularization point affects variability of the final estimate. The flow on from this, in the context of the initial motivating question, was that a relationship between asymptotic and finite-model order based quantifications was exposed.

Namely, if the regularizing point is such that excess pole-zero cancellations are constrained to be at the origin, then that part of the variance quantification due to this regularization choice will, as the model order increases well beyond the underlying true one, dominate the variance in such as way as to make it arbitrarily close to pre-existing variance expressions.

However, other choices of regularization point, implying nonorigin pole-zero cancellations, will destroy this relationship.

It is believed that this analysis can contribute to an understanding of how best to quantify variance error according to the estimation setup being employed.

Appendix A

A. Proof of Theorem 3.1

Proof: First, by the definition of $\widehat{\theta}^n_N(\delta)$

$$\left.\frac{dV_N(\theta^n,\delta)}{d\theta^n}\right|_{\theta^n=\widehat{\theta}^n_N(\delta)} = 0 \quad \text{w.p.1.} \tag{A.1}$$

Now, choose some $\theta^n_\circ \in \overline{\Theta}$. Then using the same Taylor expansion argument as employed in [3] around (A.14) of that paper

$$\frac{dV_N\left(\theta^n_\circ,\delta\right)}{d\theta^n} = R_N(\beta_N,\delta)\left(\theta^n_\circ - \widehat{\theta}^n_N(\delta)\right) \quad \text{w.p.1} \tag{A.2}$$

where

$$R_N(\beta_N,\delta) \triangleq \left.\frac{d^2V_N(\theta^n,\theta^n)}{d\theta^n d(\theta^n)^T}\right|_{\theta^n=\beta_N}$$
$$\lim_{N\to\infty}\|\beta_N - \theta^n_\circ\| = 0 \quad \text{w.p.1.} \tag{A.3}$$

Furthermore, as established in [1], [8], and [10], for any $\theta^n_\circ \in \overline{\Theta}$

$$\sqrt{N}\frac{dV_N\left(\theta^n_\circ,\delta\right)}{d\theta^n} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\sigma^2 T_n\right) \text{ as } N\to\infty \tag{A.4}$$

where, under the assumption of $\varepsilon_t\left(\theta^n_\circ\right) = e_t$ and with $\Psi(z,\theta^n)$ defined via (15), by Parseval's Theorem

$$\begin{aligned} T_n &= \lim_{N\to\infty}\mathbf{E}\left\{\frac{d}{d\theta^n}V_N\left(\theta^n_\circ\delta\right)\left[\frac{d}{d\theta^n}V_N\left(\theta^n_\circ,\delta\right)\right]^T\right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi\left(e^{j\omega},\theta^n_\circ\right)\Psi^\star\left(e^{j\omega},\theta^n_\circ\right)d\omega. \end{aligned} \tag{A.5}$$

Note that, by virtue of the evaluation of $dV_n(\theta^n,\delta)/d\theta^n$ at $\theta^n = \theta^n_\circ$, then T_n defined in (I.5) is not dependent on δ. Furthermore, as established in [6, Lemma 5.1], this matrix is singular if the model order is greater than an underlying true one. In this case, T_n will have a spectral decomposition

$$T_n = [V_1,V_2]\begin{bmatrix} S_1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T\end{bmatrix} = V_1S_1V_1^T \tag{A.6}$$

where S_1 is a diagonal matrix formed from the nonzero eigenvalues of T_n and $V_1^TV_1 = I$. This allows the definition of the pseudo-inverse $T_n^\dagger$ as

$$T_n^\dagger \triangleq V_1S_1^{-1}V_1^T. \tag{A.7}$$

In this case, (I.4) implies that as $N\to\infty$

$$\begin{aligned} &\sqrt{N}\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger\frac{dV_N\left(\theta^n_\circ,\delta\right)}{d\theta^n} \\ &\quad\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^2\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger\overline{\Psi\left(e^{j\omega},\theta^n_\circ\right)}\right). \end{aligned} \tag{A.8}$$

Additionally, combining the two equations in (I.3), and as established in [8] and [10]

$$\lim_{N\to\infty}R_N(\beta_N,\delta) = \lim_{N\to\infty}\mathbf{E}\left\{R_N\left(\theta^n_\circ,\delta\right)\right\} = T_n + \delta I \tag{A.9}$$

element-wise and with probability one. Using this formulation

$$\begin{aligned} \lim_{N\to\infty}&\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger\mathbf{E}\left\{R_N\left(\theta^n_\circ,\delta\right)\right\} \\ &= \Psi^T\left(e^{j\omega},\theta^n_\circ\right)\left[T_n^\dagger T_n + \delta T_n^\dagger\right] \\ &= \Psi^T\left(e^{j\omega},\theta^n_\circ\right) + \delta\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger. \end{aligned} \tag{A.10}$$

In progressing to the last equality, it has been recognized that, as established in the proof of [6, Lemma 5.1], a vector x is in the kernel of T_n if, and only if

$$x^\star\Psi\left(e^{j\omega},\theta^n_\circ\right) = 0, \qquad \omega\in[-\pi,\pi]. \tag{A.11}$$

Therefore, $\Psi\left(e^{j\omega},\theta^n_\circ\right)$ is orthogonal to this kernel for all ω and hence $\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger T_n = \Psi^T\left(e^{j\omega},\theta^n_\circ\right)$.

Combining (I.2), (I.8), (I.10), and the aforementioned fact that $R_N(\beta_N,\delta)\to T_n+\delta I$ with probability one implies that the following convergence in distribution holds:

$$\begin{aligned} \lim_{\delta\to 0}\lim_{N\to\infty}&\sqrt{N}H\left(e^{j\omega},\theta^n_\circ\right)S_{\zeta_\circ}^{-T}(e^{j\omega})\Psi^T\left(e^{j\omega},\theta^n_\circ\right) \\ &\times\left(\theta^n_\circ - \widehat{\theta}^n_N(\delta)\right) = \mathcal{N}\left(0,\Delta_n(\omega)\right) \end{aligned} \tag{A.12}$$

where

$$\begin{aligned} \Delta_n(\omega) = \sigma^2&\left|H\left(e^{j\omega},\theta^n_\circ\right)\right|^2 S_{\zeta_\circ}^{-\star}(e^{j\omega})\Psi^T\left(e^{j\omega},\theta^n_\circ\right) \\ &\times T_n^\dagger\overline{\Psi\left(e^{j\omega},\theta^n_\circ\right)}S_{\zeta_\circ}(e^{j\omega}). \end{aligned} \tag{A.13}$$

Finally, with e_k being the vector of all zeros save for a 1 in the k'th position, and with $\alpha\in\mathbf{C}^p$ arbitrary

$$\begin{aligned} &\left\langle\Psi^T\left(e^{j\lambda},\theta^n_\circ\right)e_k,\Psi^T\left(e^{j\lambda},\theta^n_\circ\right)T_n^\dagger\overline{\Psi\left(e^{j\omega},\theta^n_\circ\right)}\alpha\right\rangle \\ &\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi}\alpha^\star\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger\overline{\Psi\left(e^{j\lambda},\theta^n_\circ\right)}\Psi^T \\ &\quad\quad\times\left(e^{j\lambda},\theta^n_\circ\right)e_k\,d\lambda \\ &\quad= \alpha^\star\Psi^T\left(e^{j\omega},\theta^n_\circ\right)T_n^\dagger T_ne_k \\ &\quad= \alpha^\star\Psi^T\left(e^{j\omega},\theta^n_\circ\right)e_k. \end{aligned} \tag{A.14}$$

Therefore, by the same argument as used the proof of [6, Lemma 3.1], $\Psi^T\left(e^{j\lambda},\theta^n_\circ\right)T_n^\dagger\overline{\Psi\left(e^{j\omega},\theta^n_\circ\right)}$ is equal to $\varphi_n(\lambda,\omega)$, the reproducing kernel for space spanned by the columns of $\Psi^T\left(z,\theta^n_\circ\right)$. Using the same Taylor expansion argument as employed in the proof of Lemma 3.1 then completes the proof. □

$$X_n = \mathrm{Span}\left\{\left[\frac{B(z)F(z)z^{-1}}{H(z,\theta_\circ^n)A^2(z)T_G(z)}, 0\right]^T, \dots, \left[\frac{B(z)F(z)z^{-m_a}}{H(z,\theta_\circ^n)A^2(z)T_G(z)}, 0\right]^T\right\}$$
$$\oplus \mathrm{Span}\left\{\left[\frac{F(z)z^{-1}}{H(z,\theta_\circ^n)A(z)T_G(z)}, 0\right]^T, \dots, \left[\frac{F(z)z^{-m_b}}{H(z,\theta_\circ^n)A(z)T_G(z)}, 0\right]^T\right\}$$
$$\oplus \mathrm{Span}\left\{\left[0, \frac{z^{-1}}{D(z)T_H(z)}\right]^T, \dots, \left[0, \frac{z^{-m_d}}{D(z)T_H(z)}\right]^T, \left[0, \frac{z^{-1}}{C(z)T_H(z)}\right]^T, \dots, \left[0, \frac{z^{-m_c}}{C(z)T_H(z)}\right]^T\right\}.$$

APPENDIX B

A. *Proof of Corollary 3.1*

Proof: Since the conditions of Theorem 3.1 are satisfied, then the asymptotic in N co-variance is given by (21). Furthermore, in the case that a Box-Jenkins model structure is employed, then under the assumptions (27)

$$\frac{d}{dc_k}G(z,\theta^n) = \frac{d}{dd_k}G(z,\theta^n) = \frac{d}{da_k}H(z,\theta^n) = \frac{d}{db_k}H(z,\theta^n) = 0 \tag{B.15}$$

and

$$\frac{d}{da_k}G(z,\theta^n) = -\frac{B(z)}{A^2(z)T_G(z)} \cdot z^{-k}$$
$$\frac{d}{db_\ell}G(z,\theta^n) = \frac{z^{-\ell}}{A(z)T_G(z)} \tag{B.16}$$
$$\frac{dH(z,\theta^n)}{dd_k} = -\frac{C(z)z^{-k}}{D^2(z)T_H(z)}$$
$$\frac{dH(z,\theta^n)}{dc_\ell} = \frac{z^{-\ell}}{D(z)T_H(z))}. \tag{B.17}$$

Therefore, since under the assumption of $\Phi_{ue}(\omega) = 0$

$$S_\zeta(z) = \begin{bmatrix} F(z) & 0 \\ 0 & \sigma \end{bmatrix} \tag{B.18}$$

then, according to (15) and (16), the equation at the top of the page holds. Furthermore, under the assumption (28) the space X_n may be reformulated as

$$X_n = \mathrm{Span}\left\{f_1(z), \dots, f_{m_a+m_b-m_{t_g}}(z), g_1(z), \dots, g_{m_c+m_d-m_{t_h}}(z)\right\} \tag{B.19}$$

$$f_k(z) \triangleq \left[\frac{z^{-k}}{A_\dagger(z)}, \quad 0\right]^T$$
$$g_k(z) \triangleq \left[0, \quad \frac{z^{-k}}{C(z)D(z)T_H(z)}\right]^T. \tag{B.20}$$

Therefore, by [6, Lemma 3.4], the multivariable reproducing kernel for X_n is of the form

$$\varphi_n(\lambda,\mu) = \begin{bmatrix} \varphi_n^f(\lambda,\omega) & 0 \\ 0 & \varphi_m^g(\lambda,\omega) \end{bmatrix} \tag{B.21}$$

where all the entries in the above matrix are scalar. In this case, [6, Lemmas 3.1 and 3.2] may be used to quantify $\varphi_n^f(\lambda,\omega), \varphi_n^g(\lambda,\omega)$ Setting $\kappa(\omega) = \varphi_n^f(\omega,\omega)$, $\widetilde{\kappa}(\omega) = \varphi_n^g(\omega,\omega)$ completes the proof. □

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Simultaneous Tracking and Stabilization of Mobile Robots: An Adaptive Approach

K. D. Do, Z. P. Jiang, and J. Pan

***Abstract*—This note presents a time-varying global adaptive controller at the torque level that simultaneously solves both tracking and stabilization for mobile robots with unknown kinematic and dynamic parameters. The controller synthesis is based on Lyapunov's direct method and backstepping technique. Simulations illustrate the effectiveness of the proposed controller.**

***Index Terms*—Global adaptive control, Lyapunov design, mobile robot, stabilization, tracking.**

I. INTRODUCTION

The main difficulty of solving stabilization and tracking control of mobile robots is because the motion of the systems in question has more degrees of freedom than the number of inputs under nonholonomic constraints. Furthermore, the necessary condition of Brockett's

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