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Fundamental Performance Limitations of Modulated and Demodulated Control Systems

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Abstract— **We consider feedback performance limitations for modulated and demodulated control systems whose base systems have non-minimum phase (NMP) zeros or unstable poles. We first derive a transfer function for the modulated system and then show how the poles and zeros of this function are related to those of the base system. We next analyse the behaviour of the poles and zeros, when the modulation frequency is varied. Bode and Poisson Integral constraints for the modulated system are then considered. The effect of a base system delay is also discussed.**

Keywords: Performance limitations, linear systems, control applications, poles and zeros

I. Introduction

It has been well documented that open loop unstable poles, non-minimum phase (NMP) zeros and delays imply various constraints on the achievable closed loop performance for linear feedback control systems. Detailed discussions on time and frequency domain integral constraints, their relationship, and implications for controller design may be found in [1] and [2].

In this paper, we consider feedback control of modulated and demodulated systems of the type shown in Fig. 1. Here, $G(s)$ denotes the transfer function of a linear system and $d_0(t)$ represents an output disturbance. The input to $G(s)$ is $\cos \omega_0 t$ modulated (i.e., multiplied) by $u(t)$. The output is demodulated by correlating it with $\cos(\omega_0 t + \phi)$ (where ϕ is an appropriate phase shift) and passing the resulting signal through a low pass filter $F(s)$. We refer to $G(s)$ as the base system.

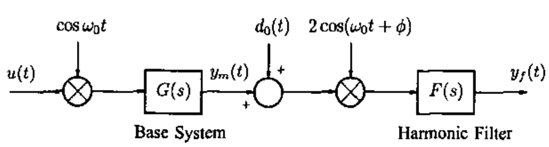

Fig. 1. Block diagram of modulated and demodulated system

Modulated and demodulated control systems are met in certain specific applications. An early example of a modulated control system is the 'envelope feedback for a radio frequency transmitter' discussed in [3, Sect. 19.3]. More recent examples of modulated and demodulated systems include vibratory microgyroscopes, such as those described in [4] and [5], and rotating gravity gradiometers[1]. The drive control loop for the gyroscope described in [9] provides a motivating example for the study of modulated control systems. The purpose of this loop is to maintain an oscillation at the resonant frequency of the device. It has been shown [10] that the automatic gain control (AGC) scheme used to achieve this is an example of a modulated and demodulated control system. We return to this example in our discussion of delays in Sect. V-A.

Our focus in the current paper is on feedback performance trade-offs for modulated and demodulated systems. In particular, we will be concerned with the limitations imposed by open right half plane (ORHP) poles and zeros of $G(s)$. The effect of a base system time delay is also considered. In Sect. II, we derive a (approximate) transfer function for the system in Fig. 1, and in Sect. III, we give example time responses. Sect. IV contains results on the behaviour (as ω_0 is varied) of the poles and zeros of this transfer function. In Sect. V, we use these results in the analysis of closed loop performance limitations for modulated systems. We pay particular attention to the implications of Bode and Poisson type integral formulae for these systems.

A. Notation

In this paper, $\arg z$ denotes the argument and $\operatorname{Arg} z$ denotes the principal argument of z. Thus, $-\pi < \operatorname{Arg} z \leq \pi$. $f(x_0^+)$ is used to denote $\lim_{x \to x_0^+} f(x)$. $f(x_0^-)$ is defined similarly. Upper case is often used to denote the Laplace transform of a signal.

II. General System Description

We return to the modulated and demodulated system shown in Fig. 1. We note that ϕ is a function of ω_0 defined by $\phi(\omega_0) = \operatorname{Arg}[G(j\omega_0)]$. However, we omit the argument of ϕ when it is clear from the context.

The following assumptions are made:

Assumptions

1) $u(t)$ is a band-limited signal having bandwidth ω_b rad/s (by this we mean that $|U(j\omega)|$ is small for $\omega > \omega_b$).
2) $\omega_0 > \omega_b$.
3) $j\omega_0$ is not a pole or zero of $G(s)$ (i.e., $\phi(\omega_0)$ is well defined).
4) $F(s)$ is a low pass filter which rolls off between ω_b and $2\omega_0 - \omega_b$.
5) $F(s)$ has no poles or zeros in the closed right half plane (CRHP).

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[1]See, for example, [6], [7], [8]. A description of the Bell rotating gradiometer can also be found at http://www.bellgeo.com under the heading 'FTG'.

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Note that the role of $F(s)$ is to significantly reduce the demodulated output components appearing at the base frequencies shifted by $2\omega_0$ relative to the base frequencies.

For any u which stabilises the modulated system, it is readily seen that $Y_f(s)$ is given by

$$F(s)\left[G_m(s,\omega_0)U(s) + \frac{1}{2}\left[e^{-j\phi}G(s+j\omega_0)U(s+2j\omega_0) + e^{+j\phi}G(s-j\omega_0)U(s-2j\omega_0)\right]\right] + D_f(s),$$

where

$$G_m(s,\omega_0) = \frac{1}{2}\left(e^{-j\phi}G(s+j\omega_0) + e^{j\phi}G(s-j\omega_0)\right)$$

and $D_f(s) = (e^{-j\phi}D_0(s+j\omega_0) + e^{j\phi}D_0(s-j\omega_0))F(s)$.

We note that Assumptions 1, 2 and 4 imply that $F(j\omega)U(j\omega \pm 2j\omega_0) \approx 0$, and so we can safely approximate the output response as

$$y_f(t) \approx \mathscr{L}^{-1}\{U(s)G_m(s,\omega_0)F(s) + D_f(s)\}.$$

It follows that the modulated system has an approximate transfer function of $G_m(s,\omega_0)F(s)$. It is clear that the fidelity of this model for the modulated system will depend on the fidelity of the base system model G at the frequencies between $\omega_0 - \omega_b$ and $\omega_0 + \omega_b$ (i.e., the baseband shifted by ω_0).

The following example clarifies the relationship between y_m, y_f and G_m.

III. Example Time Responses

Consider the following base system:

$$G(s) = \frac{s-1}{(s+5)(s+10)}.$$

Suppose that this system is modulated at $\omega_0 = 7$ rad/s. Then

$$G_m(s,7) = \frac{\cos(0.15)(s^2+1.71s+66.94)(s+11.22)}{(s^2+10s+74)(s^2+20s+149)}.$$

Let $F(s) = \dfrac{89.13}{s^4+8.03s^3+32.23s^2+75.80s+89.13}$

($F(s)$ is a fourth order Butterworth filter with a cutoff frequency of approximately 3.1 rad/s) and let $U(s) = F(s)/s$.

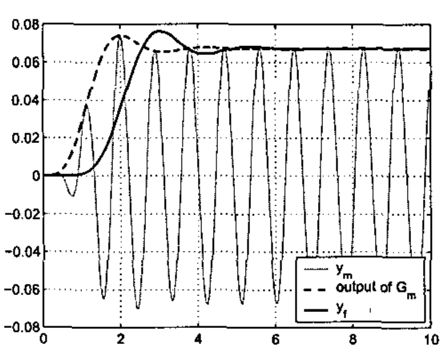

Fig. 2. Step response (modulated plant output y_m)

Fig. 2 contains plots of the modulated output $y_m(t)$ and the filtered output $y_f(t)$ the system in Fig. 1. The output of G_m (i.e. $\mathscr{L}^{-1}\{G_m(s,7)U(s)\}$) is also shown. It can be seen that the output of G_m is the envelope of $y_m(t)$, and that $y_f(t)$ is an approximation of the envelope filtered by $F(s)$. We note that the slight 'delay' observed in $y_f(t)$ relative to the output of G_m is due to the phase shift of the low pass filter.

IV. Poles, Zeros, and Delays

Since $F(s)$ is assumed to be stable and minimum phase, the feedback performance limitations of the modulated system are determined by $G_m(s,\omega_0)$, especially its poles, zeros and delay. In this section, we analyse the behaviour of the poles, zeros and delay of $G_m(s,\omega_0)$ as functions of the modulation frequency ω_0. We note that this section is a condensed version of [10, Sect. 4]. Proofs of the results and illustrative examples are given in [10].

Suppose that $G(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are polynomials with real coefficients. We assume that $N(s)$ and $D(s)$ are coprime and can be written as $N(s) = \prod_{i=1}^{m}(s-z_i)$ and $D(s) = \prod_{i=1}^{n}(s-p_i)$, where $z_i,\ p_i \in \mathbf{C}$ and $\mathrm{Re}[z_i] \neq 0$. We also assume that $r = n - m > 0$, i.e., that $G(s)$ is strictly proper. Then

$$G_m(s,\omega_0) = \frac{1}{2}\frac{N_m(s,\omega_0)}{D_m(s,\omega_0)}, \tag{1}$$

where $N_m(s,\omega_0) = e^{-j\phi}N(s+j\omega_0)D(s-j\omega_0) + e^{j\phi}N(s-j\omega_0)D(s+j\omega_0), \quad (2)$

and $D_m(s,\omega_0) = D(s+j\omega_0)D(s-j\omega_0). \quad (3)$

We note that $N_m(s,\omega_0)$ and $D_m(s,\omega_0)$ may have common factors for some modulation frequencies. However, we will show that this occurs only at isolated values of ω_0. Hence, the zeros of $D_m(s,\omega_0)$ will, in the sequel, be referred to as the *poles of the modulated system* (or of $G_m(s,\omega_0)$). Similarly, the zeros of $N_m(s,\omega_0)$ will be referred to as the *zeros of* $G_m(s,\omega_0)$.

A. Poles

An immediate consequence of (3) is the following:

Lemma IV.1 *For each $\omega_0 \in \mathbf{R}$, the zeros of $D_m(s,\omega_0)$ are given by $s = p_i \pm j\omega_0$ for $i = 1,\ldots,n$.*

Remark 1 We thus see that the poles of the transfer function $G_m(s,\omega_0)$ are simply shifted forms of the poles of $G(s)$. This is a straightforward connection. □

B. Zeros

Determining the zeros of $N_m(s,\omega_0)$ is, in general, more difficult.[2] We can, however, gain some insight into the location of the zeros by analysing the limiting behaviour of the zeros as $\omega_0 \to 0$ and as $\omega_0 \to \infty$. We first note

[2] We note that the system in Fig. 1 is a periodic system. Hence, the relative degree of the system can be determined from [11, Def. 3]. However, the results in [11] on computing zeros cannot be applied to this system because it does not have a uniform relative degree.

that, for a given ω_0, $N_m(s,\omega_0)$ is a polynomial in s. Thus $N_m(s,\omega_0)$ can be written as

$$N_m(s,\omega_0) = \sum_{i=0}^{n+m} c_i(\omega_0)s^i. \qquad (4)$$

Since $N_m(a,\omega_0)$ is real $\forall\, a \in \mathbf{R}$, the coefficients $c_i(\omega_0)$ are real functions of ω_0. It is also clear that c_i is continuous at $\omega_0 = \omega_1$ if $j\omega_1$ is not a pole or zero of $G(s)$.

Suppose that $\forall\, \omega_0 \in (\omega_1, \omega_2)$, $j\omega_0$ is not a pole or zero of $G(s)$ and the degree of $N_m(s,\omega_0)$ is M. We let the zeros of $N_m(s,\omega_0)$ be denoted by $\zeta_i(\omega_0)$, $i = 1,...,M$. Then $N_m(s,\omega_0)$ can also be expressed in the following form:

$$N_m(s,\omega_0) = c_M(\omega_0)\prod_{i=1}^{M}(s-\zeta_i(\omega_0)). \qquad (5)$$

Since the coefficients of $N_m(s,\omega_0)$ are continuous on (ω_1,ω_2), the zeros of $N_m(s,\omega_0)$ are also continuous functions of ω_0.

Since $N(s)$ and $D(s)$ are monic, it follows from (2) that

$$c_{n+m}(\omega_0) = 2\cos\phi(\omega_0). \qquad (6)$$

Equation (6) implies that the degree of $N_m(s,\omega_0)$ will be $< n+m$ whenever $|\phi(\omega_0)| = \pi/2$. However, as stated in the following lemma, the degree cannot be $< n+m-1$.

Lemma IV.2 *For each $\omega_0 > 0$, the degree of $N_m(s,\omega_0)$ is*

$$\begin{aligned} & m+n \quad \textit{if} \quad |\phi(\omega_0)| \neq \pi/2 \\ \textit{and} \quad & m+n-1 \quad \textit{if} \quad |\phi(\omega_0)| = \pi/2. \end{aligned}$$

The following lemma describes the behaviour of ζ_i, $i = 1,...,M$ as $\omega_0 \to \omega_1^+$. We note that the lemma is stated for the case of $\omega_0 \to \omega_1^+$ but clearly also holds for the case of $\omega_0 \to \omega_1^-$.

Lemma IV.3 *Consider the polynomial (in s) defined by (4). Let M be the degree of $N_m(s,\omega_0)$ as ω_0 approaches ω_1 from above. Suppose that $c_i(\omega_1^+)$ is finite $\forall\, i$ and let $M' \leq M$ be the degree of $N_m(s,\omega_1^+)$. Then as $\omega_0 \to \omega_1^+$, M' of the zeros of $N_m(s,\omega_0)$ tend to the zeros of $N_m(s,\omega_1^+)$. If $M - M' = 1$, then the remaining zero tends to ∞ or $-\infty$.*

Lem. IV.2 and Lem. IV.3 imply that if $|\phi(\omega_0)| = \pi/2$ at $\omega_0 = \omega_1$, then $n+m-1$ of the zeros are continuous at $\omega_0 = \omega_1$ and the remaining zero tends to ∞ or $-\infty$ as $\omega_0 \to \omega_1^+$ or ω_1^-. We also note that if $G(s)$ has a pole or zero of multiplicity m_1 at $j\omega_1$, then $c_i(\omega_1^+) = -c_i(\omega_1^-)$ if m_1 is odd and $c_i(\omega_1^+) = c_i(\omega_1^-)$ if m_1 is even. Thus, provided that $c_M(\omega_1^+) \neq 0$, there exist M continuous functions $\zeta_i(\omega_0)$ which satisfy (5) in the neighbourhood of ω_1.

We are now in a position to present two important results on the zero loci of the modulated system. These describe the behaviour of the zeros as $\omega_0 \to 0$ and as $\omega_0 \to \infty$, respectively.

Theorem IV.4 (a) *Let $\omega_1 > 0$ be chosen s.t. $N_m(s,\omega_0)$ has degree M on $(0,\omega_1)$. Let μ be the number of singularities (i.e., poles or zeros of $G(s)$) at the origin, and let the sets of zeros and poles of $G(s)$ be denoted by $\mathcal{Z}_G$ and $\mathcal{P}_G$, respectively. Also let*

$$\begin{aligned} & \mathcal{Z}_0 = \{\zeta_i(0^+) : |\zeta_i(0^+)| \neq \infty, i = 1,...,M\} \\ \textit{and} \quad & \mathcal{Z}_1 = \{z_0 : N_\omega(z_0) = 0\}, \end{aligned} \qquad (7)$$

where $N_\omega(s) = \phi'(0^+)N(s)D(s) - N'(s)D(s) + N(s)D'(s)$. Then

$$\mathcal{Z}_0 = \begin{cases} \mathcal{Z}_G \cup \mathcal{P}_G, & \textit{if } \mu \textit{ is even,} \\ \mathcal{Z}_1, & \textit{if } \mu \textit{ is odd.} \end{cases}$$

(b) *Suppose that μ is even, and α is a pole or zero (of $G(s)$) of multiplicity m_α. Let $\zeta_i(0^+) = \alpha$ for $i = 1,...,m_\alpha$. Then the following limits:*

$$\lim_{\omega_0\to 0^+}\frac{\zeta_i(\omega_0)-\alpha}{\omega_0}, \quad i = 1,...,m_\alpha \qquad (8)$$

are distinct and are given by

$$\begin{cases} \tan\left(\frac{k\pi}{m_\alpha}\right), & m_\alpha \textit{ odd,} \\ \tan\left(\frac{\pi}{2m_\alpha}+\frac{k\pi}{m_\alpha}\right), & m_\alpha \textit{ even,} \end{cases} \qquad (9)$$

for $k = 1,...,m_\alpha$.

Remark 2 Let the limit (8) be denoted by ζ_i'. Thm. IV.4(b) implies that m_α (μ even) or $m_\alpha - 1$ (μ odd) of the ζ_i''s are real and non-zero. It follows that the angle of departure of each of these loci is 0 or π. If μ is odd then there is exactly one value of k s.t. $\zeta_k(0^+) = \alpha$ and $\zeta_k' = 0$. If α is real then ζ_k also has an angle of departure of 0 or π because complex zeros must occur in conjugate pairs. □

Next we consider the case $\omega_0 \to \infty$:

Theorem IV.5 *(a) Let $\eta_i(\omega_0) = \zeta_i(\omega_0)/\omega_0$ for $\omega_0 > 0$. As $\omega_0 \to \infty$, $2m$ of the zeros of $N_m(s,\omega_0)$ tend to $z_i + j\omega_0$ and $z_i - j\omega_0$, $i = 1,...,m$.*
(b) If r is even, then the remaining zeros satisfy the following condition:

$$\lim_{\omega_0\to\infty}\eta_i(\omega_0) = -\tan\left(\frac{\pi}{2r}+\frac{k\pi}{r}\right), \quad k = 0,...,r-1.$$

If r is odd, then $r-1$ of the remaining zeros satisfy the following condition:

$$\lim_{\omega_0\to\infty}\eta_i(\omega_0) = -\tan\left(\frac{\pi}{2}+\frac{k\pi}{r}\right), \quad k = 1,...,r-1,$$

and the final η_i tends to ∞ or $-\infty$.

For almost all $\omega_0 > 0$, the zeros and poles of $G_m(s,\omega_0)$ will be the same as the zeros of $N_m(s,\omega_0)$ and $D_m(s,\omega_0)$, respectively. However, at isolated values of ω_0 we may have 'pole-zero' cancellations as stated in the following lemma.

Lemma IV.6 *For each $\omega_1 > 0$, $N_m(s,\omega_1)$ and $D_m(s,\omega_1)$ have a common zero iff $\exists\, k, l \in \{1,...,n\}$ s.t.*

$$p_l = p_k + 2j\omega_1. \qquad (10)$$

Let m_i denote the multiplicity of p_i for $i = 1, ..., n$. If $\omega_1 > 0$, and condition (10) is satisfied, then $N_m(s, \omega_1)$ has at least $\min\{m_k, m_l\}$ zeros at $p_k + j\omega_1 = p_l - j\omega_1$.

Remark 3 We have thus seen that the zeros of the transfer function $G_m(s, \omega_0)$ are, in general, not simply related to the zeros of $G(s)$. However, Thm. IV.5 shows that for large ω_0 (relative to the location of the poles of $G(s)$), the zeros of $G_m(s, \omega_0)$ approach the shifted forms of the zeros of $G(s)$ together with some extra zeros which converge to specific asymptotes. □

Remark 4 The situation described in Remark 3, and formalised by Thm. IV.5, is reminiscent of the zeros of unmodulated sampled data systems having zero order hold input. We recall that, when expressed in the equivalent delta domain [12], the zeros of these systems tend, as the sampling rate is increased, to the zeros of the underlying continuous time system, together with some extra zeros (sometimes called the sampling zeros) which converge to specific locations ([13], [12]). □

C. Delays

We next consider the impact of delays in the base system. The following lemma states that if a linear system is modulated and demodulated, then the delay is preserved.

Lemma IV.7 *Suppose that $\tilde{G}(s) = e^{-s\tau}G(s), \quad \tau > 0$. Then $\tilde{G}_m(s, \omega_0) = e^{-s\tau}G_m(s, \omega_0)$.*

D. Summary

In this section, we have shown that the poles of $G_m(s, \omega_0)$ are given by $p_i \pm j\omega_0$. The behaviour of the loci of the zeros is more complex. It was found that the loci are continuous (on $\mathbf{R}^+$) except at points where $|\phi(\omega_0)|$ crosses (or touches) $\pi/2$. At these points, one of the zeros 'vanishes' and the rest are continuous. As $\omega_0 \to 0$ the zeros tend to the poles and zeros of $G(s)$ when $G(s)$ has an even number of integrators (or differentiators), and the zeros of $\phi'(0^+)N(s)D(s) - N'(s)D(s) + N(s)D'(s)$ when the number of integrators is odd. At high modulation frequencies (relative to the location of the poles and zeros of $G(s)$), $2m$ of the zeros tend to $z_i + j\omega_0$ and $z_i - j\omega_0$ and the remaining zeros tend to ∞ or $-\infty$. Finally, it was shown that the delay of a system is invariant with respect to modulation and demodulation.

V. Implications on Feedback Performance Trade-offs

The results of the previous section relate the poles and zeros of a modulated system $G_m(s, \omega_0)$ to the poles and zeros of its base system $G(s)$. In this section, we discuss the implications of these results on the closed loop control of the modulated system when $G(s)$ is nonminimum phase (NMP) or unstable (i.e., $G(s)$ has (ORHP) zeros or poles). In particular, we consider the performance limitations of the feedback system shown in Fig. 3. We replace this system by the (approximate) modulated system shown in Fig. 4. Note that in this figure, $r(t)$ is the reference signal and $C(s)$ is the transfer function of a stable, proper, minimum phase controller. For each ω_0, the loop transfer function is $L_{\omega_0}(s) = C(s)G_m(s, \omega_0)F(s)$, and the sensitivity and complementary sensitivity functions are

$$S_{\omega_0}(s) = \frac{1}{1 + L(s)} \quad \text{and} \quad T_{\omega_0}(s) = 1 - S_{\omega_0}(s),$$

respectively. We note that, since $G_m(s, \omega_0)$ and $F(s)$ are strictly proper, the relative degree of $L_{\omega_0}(s)$ is > 2.

A. Delays and Their Effect on the Closed Loop Bandwidth

It is well known that plant delays imply constraints on the achievable closed loop bandwidth. In the case of modulated systems it is worthwhile to note that the delay limits the bandwidth of the system in Fig. 4 not the system in Fig. 3. In particular, the speed of the oscillation at $y_m(t)$ (or the modulation frequency) is not limited by the delay. However, the constraint on the closed loop response at y_f implies that the speed of response of the envelope of $y_m(t)$ is constrained.

Returning briefly to the gyroscope example in the introduction, we note that in [9], the implementation of the AGC scheme introduces a controller delay of the order of 1 ms. The above discussion provides an alternative explanation for the observation (originally made in [9]) that it is possible to regulate the oscillation at f_1 (≈ 4.5 kHz) despite the large delay.

B. Impact of Zeros and Delays

Suppose that $G(s)$ has an NMP zero at z_k. If $G(s)$ has an even number (possibly zero) of singularities at the origin, then Thm. IV.4(a) implies that $G_m(s, \omega_0)$ will have an NMP zero near z_k for small ω_0. It follows that $T_{\omega_0}(s)$ will have a zero close to z_k. If $G(s)$ has an odd number of singularities at the origin then these statements hold if the multiplicity of z_k is > 1.

As $\omega_0 \to \infty$, two of the zeros of $G_m(s, \omega_0)$ will tend to $z_k + j\omega_0$ and $z_k - j\omega_0$ (Thm. IV.5). It follows that $G_m(s, \omega_0)$ has two NMP zeros when ω_0 is large. If the relative degree of $G(s)$ is > 1 then $G_m(s, \omega_0)$ will also have at least one large NMP zero on the positive real axis. We recall that, if $\zeta_{k_i}(\omega_0)$, $i = 1, ..., n_z$ are the NMP zeros of $G_m(s, \omega_0)$ and τ is the delay of the system, then the right hand side (RHS) of the Bode integral for $T_{\omega_0}(s)$ is given by [2, Eq. 3.17]

$$\frac{\pi}{2}\frac{1}{T_{\omega_0}(0)}\lim_{s\to 0}\frac{dT_{\omega_0}(s)}{ds} + \pi\sum_{i=1}^{n_z}\frac{1}{\zeta_{k_i}(\omega_0)} + \frac{\pi}{2}\tau.$$

It has been shown that as $\omega_0 \to \infty$, $G_m(s, \omega_0)$ will be NMP if $G(s)$ is NMP or $G(s)$ has a relative degree > 1. However, we note that as $\omega_0 \to \infty$, $|\zeta_i(\omega_0)| \to \infty$, and so the Bode integral constraint tends to that of a minimum phase system.

C. Impact of Poles

The poles of $G(s)$ affect both the poles and the zeros of $G_m(s, \omega_0)$. We first observe that the real parts of the poles of $G_m(s, \omega_0)$ and $G(s)$ are the same. Hence, if $G(s)$ is unstable, then $G_m(s, \omega_0)$ is also unstable (unless all

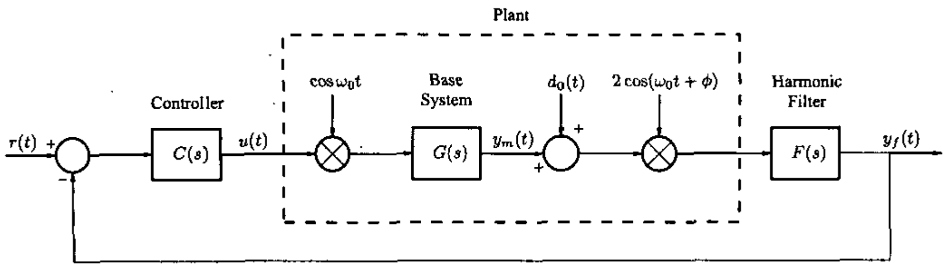

Fig. 3. Feedback control loop

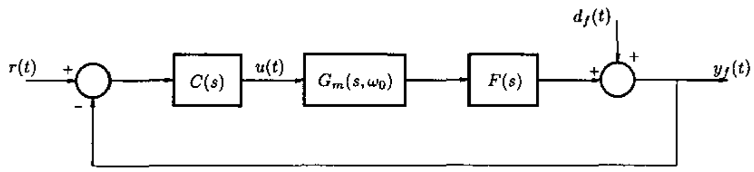

Fig. 4. Equivalent feedback control loop (ignoring high harmonics)

of the unstable poles of $1/D_m(s,\omega_0)$ are cancelled by zeros of $N_m(s,\omega_0)$). Let p_i, $i=1,...,n_p$ be the ORHP poles of $G(s)$. We note that in the absence of pole zero cancellations, the sum of the unstable poles of $G_m(s,\omega_0)$ is given by $2\sum_{i=1}^{n_p} p_i$. Since the loop transfer function has relative degree > 2, this implies that the RHS of the Bode Integral for $S_{\omega_0}(s)$ [2, Eq. 3.14] is given by $2\pi\sum_{i=1}^{n_p} p_i$. We note that this expression is independent of the modulation frequency.

Suppose that $G(s)$ has an unstable pole at p_k. Thm. IV.4 implies that, for small ω_0, the effect of p_k on the zeros of $N_m(s,\omega_0)$ is identical to that of an NMP zero. Hence the remarks on the effect NMP zeros at low modulation frequencies also hold for unstable poles.

If $G(s)$ has an even number ($\neq 0$) of integrators (or differentiators) then Thm. IV.4(b) also implies that $G_m(s,\omega_0)$ will have a small nonminimum phase zero for small ω_0.

D. Pole-Zero Interactions

Unstable poles of $G(s)$ may also cause approximate pole-zero cancellations (in the ORHP) in the modulated system. In particular, an approximate cancellation will occur when ω_0 is small and $\zeta_i(0^+) = p_k$ for some i, or when ω_0 is close to the resonant frequency of a conjugate pair of poles of $G(s)$ (Lem. IV.6). Thus, in these cases, large peaks in the closed loop sensitivity functions will be unavoidable as the RHS of the Poisson Integrals for $S_{\omega_0}(s)$ and $T_{\omega_0}(s)$ will be large [2, Thms. 3.3.1 and 3.3.2]. This implies that if the modulation frequency is 'small' relative to the unstable poles, then there will be large peaks in the sensitivity functions [2, Cors. 3.3.3 and 3.3.4]. Since the bandwidth of the closed loop is limited by the modulation frequency, this is consistent with the known result that the bandwidth should be large relative to the open loop poles.

We observe that as $\omega_0 \to \infty$, the poles and zeros of $G_m(s,\omega_0)$ are the poles and zeros of $G(s)$ shifted by $j\omega_0$ and $-j\omega_0$. $G_m(s,\omega_0)$ also has r additional zeros which tend to ∞ or $-\infty$ along the real axis (Thm. IV.5). Now suppose that $q(\omega_0)$ is a zero of $G_m(s,\omega_0)$ and that $q(\omega_0) \to z_k - j\omega_0$ or $q(\omega_0) \to \infty$. Then

$$\lim_{\omega_0\to\infty}\left|\frac{(p_k+j\omega_0)-q(\omega_0)}{(\bar{p}_k-j\omega_0)+q(\omega_0)}\right| = 1.$$

On the other hand, if $q(\omega_0) \to z_k + j\omega_0$, then

$$\lim_{\omega_0\to\infty}\left|\frac{(p_k+j\omega_0)-q(\omega_0)}{(\bar{p}_k-j\omega_0)+q(\omega_0)}\right| = \left|\frac{p_k-z_k}{\bar{p}_k+z_k}\right|.$$

It follows that as $\omega_0 \to \infty$

$$\log|B^{-1}_{S_{\omega_0}}(q(\omega_0))| \to \log|B^{-1}_S(z_k)|,$$

where $B_{S_{\omega_0}}(s)$ and $B_S(s)$ are the Blaschke products [2, Eq. (3.26)] for the modulated system and the base system, respectively. This implies that the RHS of the Poisson integral for $S_{\omega_0}(s)$ tends to that of the base system. Since it is also true that $\log|B^{-1}_{T_{\omega_0}}(p_k+j\omega_0)| \to \log|B^{-1}_T(p_k)|$, and the real parts of the poles and the delay of $G_m(s,\omega_0)$ are the same as those of $G(s)$, the RHS of the Poisson integral for $T_{\omega_0}(s)$ also tends to that of the base system.

We note that if $q(\omega_0) = z_k + j\omega_0$, then as ω_0 is increased, the peak in the weighting function in the Poisson Integral for $S_{\omega_0}(s)$ shifts to a higher frequency. This implies that, at high modulation frequencies, if $\log|B^{-1}_{S_{\omega_0}}(z_k+j\omega_0)| \approx \log|B^{-1}_S(z_k)|$, then the lower bound on the peak sensitivity (given in Cor. 3.3.3 of [2]) decreases as ω_0 is increased. In a similar manner, it can be shown that the lower bound on the peak complementary sensitivity increases as ω_0 is increased.

We illustrate these ideas by a simple example.

Example 1

Let

$$G(s) = \frac{s-5}{(s-0.2+0.2j)(s-0.2-0.2j)}.$$

In this case, $G_m(s,\omega_0)$ has four poles for $\omega_0 > 0$ and for $\omega_0 \neq \omega_x \approx 0.29$ it has three zeros. At $\omega_0 = \omega_x$, $G_m(s,\omega_0)$

has only two zeros. All of the poles are unstable, and two of the zeros are NMP. The third zero is NMP for $\omega_0 < \omega_x$ and minimum phase for $\omega_0 > \omega_x$. At $\omega_0 = 0.2$, there is an unstable (ORHP) pole-zero cancellation at $s = 0.2$. The RHS of the Poisson integrals for $S_{\omega_0}(s)$ and $T_{\omega_0}(s)$ are plotted against ω_0 in Figs. 5 and 6, respectively. We note that in Fig. 5 the number of curves changes from three to two at $\omega_0 \approx \omega_x$. Fig. 7 contains plots of the lower bounds on $\|S_{\omega_0}\|_\infty$ and $\|T_{\omega_0}\|_\infty$ (where $\|\cdot\|$ denotes the ∞ norm).

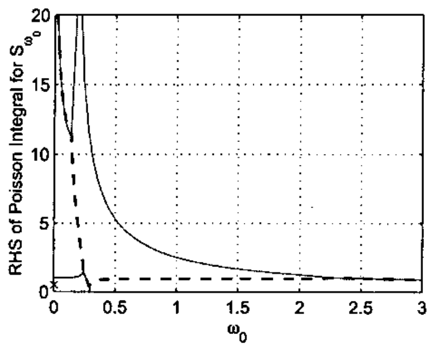

Fig. 5. Example 3 - RHS of Poisson integral for $S_{\omega_0}(s)$ evaluated at each NMP zero

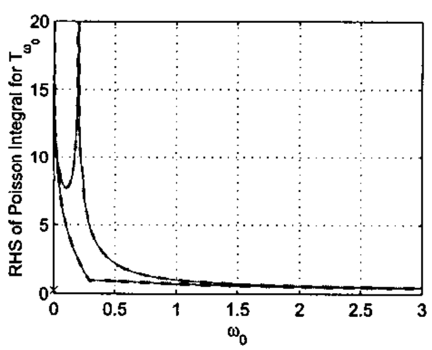

Fig. 6. Example 3 - RHS of Poisson integral for $T_{\omega_0}(s)$ evaluated at each unstable pole

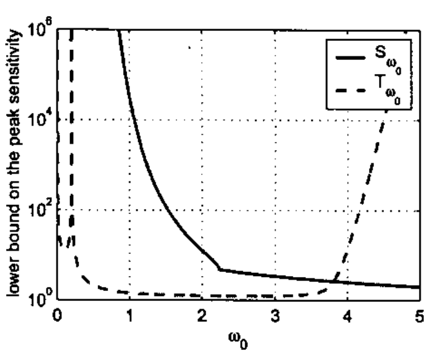

Fig. 7. Example 3 - Lower bounds on the peak sensitivities.

These bounds are obtained by letting $\alpha_1 = \alpha_2 = 1/2$ and $\omega_1 = \omega_2 = 4$ in Cors. 3.3.3 and 3.3.4 of [2]. In Figs. 5 to 7, the effect of the approximate pole-zero cancellations near $\omega_0 = 0$ and $\omega_0 = 0.2$ is clearly visible. It can also be seen that for large ω_0, the lower bound on the peak sensitivity is decreasing whilst that of the complementary sensitivity is increasing. By plotting over a larger range of ω_0, it is also possible to verify that as $\omega_0 \to \infty$, two of the curves in Fig. 5 approach $\pi \log |B_S^{-1}(5)| \approx 0.5$ and all four curves in Fig. 6 approach $\pi \log |B_T^{-1}(0.2 \pm 0.2j)| \approx 0.25$.

VI. Conclusion

In this paper, the poles, zeros and delays of modulated and demodulated systems have been analysed. It has been shown that the poles of the modulated system (G_m) are those of the base system (G) shifted by $\pm j\omega_0$ and that the delay is preserved. Several results on the continuity and asymptotic behaviour of the zero loci have also been given. The closed loop performance limitations of modulated systems whose base systems have ORHP poles or zeros were then discussed. It has been observed that G_m is unstable if (and only if) G is unstable. Also, if G has NMP zeros, then G_m has NMP zeros when the modulation frequency is very low or very high (relative to the location of the poles and zeros of the base system). Unstable poles of G also result in NMP zeros at low modulation frequencies. These zeros are particularly problematic as they may result in approximate ORHP pole-zero cancellations, and hence large peaks in the sensitivity functions will be unavoidable.

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