BINOMIAL SUMS RELATED TO RATIONAL APPROXIMATIONS TO $\zeta(4)$

Wadim Zudilin[‡] (Moscow)

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Abstract. For the solution $\{u_n\}_{n=0}^\infty$ to the polynomial recursion $(n+1)^5u_{n+1} - 3(2n+1)(3n^2+3n+1)(15n^2+15n+4)u_n - 3n^3(3n-1)(3n+1)u_{n-1} = 0$, where $n=1,2,\dots$, with the initial data $u_0=1$, $u_1=12$, we prove that all u_n are integers. The numbers u_n, $n=0,1,2,\dots$, are denominators of rational approximations to $\zeta(4)$ (see math.NT/0201024). We use Andrews's generalization of Whipple's transformation of a terminating ${}_7F_6(1)$-series and the method from math.NT/0311114.

Consider the following 3-term polynomial recursion:

$$\begin{aligned}(n+1)^5u_{n+1} &- 3(2n+1)(3n^2+3n+1)(15n^2+15n+4)u_n\\ &- 3n^3(3n-1)(3n+1)u_{n-1} = 0 \qquad \text{for} \quad n\geqslant 1,\end{aligned}$$

and take the two linearly independent solutions $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ determined by the inicial conditions $u_0=1$, $u_1=12$ and $v_0=0$, $v_1=13$. In [Z1], we give a hypergeometric interpretation of the sequence $u_n\zeta(4)-v_n$, $n=0,1,2,\dots$, from which one obtains the limit

$$\lim_{n\to\infty}\frac{v_n}{u_n} = \zeta(4) = \frac{\pi^4}{90}$$

and the representation

$$\begin{aligned}u_n &= (-1)^{n+1}\sum_{l=0}^{n}\frac{\mathrm{d}}{\mathrm{d}l}\Bigl(\frac{n}{2}-l\Bigr)\binom{n}{l}^4\binom{n+l}{n}^2\binom{2n-l}{n}^2\\ &= (-1)^n\sum_{l=0}^{n}\Bigl(\frac{n}{2}-l\Bigr)\binom{n}{l}^4\binom{n+l}{n}^2\binom{2n-l}{n}^2\\ &\qquad\times\biggl(\frac{1}{n/2-l} - 6H_{n-l} + 6H_l - 2H_{n+l} + 2H_{2n-l}\biggr),\end{aligned} \tag{1}$$

where $H_l=\sum_{j=1}^{l} j^{-1}$ are harmonic numbers. The integrality of all u_n (conjectured in [Z1]) is not an immediate consequence of formula (1). In the recent work [KR]

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C. Krattenthaler and T. Rivoal prove (among several other useful theorems and beautiful binomial identities) that

$$u_n=\sum_{i,j}\binom{n}{i}^2\binom{n}{j}^2\binom{n+j}{n}\binom{n+j-i}{n}\binom{2n-i}{n}, \qquad n=0,1,2,\dots,$$

from which one has the desired inclusions $u_n\in\mathbb{Z}$. The main objective of the present note is to give a simpler proof of the formula for the numbers u_n as well as to indicate some other representations that also show that all u_n are integers.

We use the standard notation

$${}_{r+1}F_r\left(\begin{matrix} a_0,a_1,\dots,a_r \\ b_1,\dots,b_r\end{matrix}\,\middle|\, z\right)=\sum_{l=0}^{\infty}\frac{(a_0)_l(a_1)_l\cdots(a_r)_l}{l!\,(b_1)_l\cdots(b_r)_l}z^l$$

for the generalized hypergeometric series; the notation $(a)_l=a(a+1)\cdots(a+l-1)$ for $l=1,2,\dots$ and $(a)_0=1$ stands for the Pochhammer symbol.

The following formula is due G. E. Andrews. Making the passage $q\to1$ in [A, Theorem 4] (see also [Z2] for a related application of the identity) we have: *for $s\geqslant1$ and m a non-negative integer*,

$$\begin{aligned}
&{}_{2s+3}F_{2s+2}\left(\begin{matrix} a,1+\frac12a, & b_1, & c_1, & b_2, & c_2, & \dots \\ \frac12a, & 1+a-b_1, & 1+a-c_1, & 1+a-b_2, & 1+a-c_2, & \dots\end{matrix}\right.\\
&\qquad\qquad\qquad\left.\begin{matrix} \dots, & b_s, & c_s, & -m \\ \dots, & 1+a-b_s, & 1+a-c_s, & 1+a+m\end{matrix}\,\middle|\,1\right)\\
&\quad=\frac{(1+a)_m(1+a-b_s-c_s)_m}{(1+a-b_s)_m(1+a-c_s)_m}\sum_{l_1\geqslant0}\frac{(1+a-b_1-c_1)_{l_1}(b_2)_{l_1}(c_2)_{l_1}}{l_1!\,(1+a-b_1)_{l_1}(1+a-c_1)_{l_1}}\\
&\qquad\times\sum_{l_2\geqslant0}\frac{(1+a-b_2-c_2)_{l_2}(b_3)_{l_1+l_2}(c_3)_{l_1+l_2}}{l_2!\,(1+a-b_2)_{l_1+l_2}(1+a-c_2)_{l_1+l_2}}\cdots\\
&\qquad\times\sum_{l_{s-1}\geqslant0}\frac{(1+a-b_{s-1}-c_{s-1})_{l_{s-1}}(b_s)_{l_1+\dots+l_{s-1}}(c_s)_{l_1+\dots+l_{s-1}}}{l_{s-1}!\,(1+a-b_{s-1})_{l_1+\dots+l_{s-1}}(1+a-c_{s-1})_{l_1+\dots+l_{s-1}}}\\
&\qquad\quad\times\frac{(-m)_{l_1+\dots+l_{s-1}}}{(b_s+c_s-a-m)_{l_1+\dots+l_{s-1}}}.
\end{aligned}$$

Taking $s=3$, $a=-n-2\varepsilon$, $b_1=b_2=b_3=c_2=-n-\varepsilon$, $c_1=c_3=n-\varepsilon+1$ and $m=n$, $i=l_1$, $j=l_1+l_2$, we derive from Andrews's formula

$$\begin{aligned}
&\sum_{l=0}^{n}\frac{-\frac n2-\varepsilon+l}{-\frac n2-\varepsilon}\cdot\frac{(-n-2\varepsilon)_l}{(1)_l}\cdot\frac{(-n)_l}{(1-2\varepsilon)_l}\cdot\left(\frac{(1+n-\varepsilon)_l}{(-2n-\varepsilon)_l}\right)^2\cdot\left(\frac{(-n-\varepsilon)_l}{(1-\varepsilon)_l}\right)^4\\
&\quad=\frac{(1-n-2\varepsilon)_n(-n)_n}{(1-\varepsilon)_n(-2n-\varepsilon)_n}\sum_i\frac{(-n)_i(-n-\varepsilon)_i^2}{i!\,(1-\varepsilon)_i(-2n-\varepsilon)_i}\\
&\qquad\times\sum_j\frac{(1+n)_{j-i}(-n-\varepsilon)_j(1+n-\varepsilon)_j(-n)_j}{(j-i)!\,(1-\varepsilon)_j^2j!}.
\end{aligned}\tag{2}$$

Using the trivial equality

$$(1-n-2\varepsilon)_n = \frac{-\varepsilon}{-\frac{n}{2}-\varepsilon}\cdot(-n-2\varepsilon)_n,$$

we may rewrite (2) in the form

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_{l=0}^{n} A_l(\varepsilon)\\
&\quad=\frac{1}{\varepsilon}\sum_{l=0}^{n}\Big(\frac{n}{2}+\varepsilon-l\Big)\cdot\frac{(-n-2\varepsilon)_l}{(1)_l}\cdot\frac{(-n)_l}{(1-2\varepsilon)_l}\cdot\bigg(\frac{(1+n-\varepsilon)_l}{(-2n-\varepsilon)_l}\bigg)^2\cdot\bigg(\frac{(-n-\varepsilon)_l}{(1-\varepsilon)_l}\bigg)^4\\
&\quad=\frac{(1-n-2\varepsilon)_n(-n)_n}{(1-\varepsilon)_n(-2n-\varepsilon)_n}\sum_i\frac{(-n)_i(-n-\varepsilon)_i^2}{i!\,(1-\varepsilon)_i(-2n-\varepsilon)_i}\\
&\qquad\quad\times\sum_j\frac{(1+n)_{j-i}(-n-\varepsilon)_j(1+n-\varepsilon)_j(-n)_j}{(j-i)!\,(1-\varepsilon)_j^2 j!}.
\end{aligned}\tag{3}$$

Now, we tend ε to 0. On the right hand side of (3) we only need to plug $\varepsilon=0$. To proceed with the left hand side, we first note that $A_l(0)=-A_{n-l}(0)$ for all $l=0,1,\ldots,n$, hence

$$\lim_{\varepsilon\to 0}\sum_{l=0}^{n}A_l(\varepsilon)=\sum_{l=0}^{n}A_l(0)=0\tag{4}$$

and we may apply the l'Hôpital rule:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{l=0}^{n}A_l(\varepsilon)=\sum_{l=0}^{n}\frac{\partial A_l(\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}=\sum_{l=0}^{n}A_l(0)\cdot\frac{\partial\operatorname{Log}A_l(\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}\\
&\quad=\sum_{l=0}^{n}A_l(0)\cdot\bigg(\frac{1}{\frac{n}{2}-l}-2\sum_{j=1}^{l}\frac{1}{-n+j-1}+2\sum_{j=1}^{l}\frac{1}{j}-2\sum_{j=1}^{l}\frac{1}{n+j}\\
&\qquad\quad+2\sum_{j=1}^{l}\frac{1}{-2n+j-1}-4\sum_{j=1}^{l}\frac{1}{-n+j-1}+4\sum_{j=1}^{l}\frac{1}{j}\bigg)\\
&\quad=\sum_{l=0}^{n}A_l(0)\cdot\bigg(\frac{1}{\frac{n}{2}-l}+6(H_n-H_{n-l})+6H_l-2(H_{n+l}-H_n)-2(H_{2n}-H_{2n-l})\bigg)\\
&\quad=\sum_{l=0}^{n}A_l(0)\cdot\bigg(\frac{1}{\frac{n}{2}-l}-6H_{n-l}+6H_l-2H_{n+l}+2H_{2n-l}\bigg),
\end{aligned}$$

where on the last step we use the following consequences of (4):

$$\sum_{l=0}^{n}A_l(0)H_n=\sum_{l=0}^{n}A_l(0)H_{2n}=0.$$

Since

$$A_l(0)\cdot\bigg(\frac{1}{\frac{n}{2}-l}-6H_{n-l}+6H_l-2H_{n+l}+2H_{2n-l}\bigg)=-\frac{\mathrm{d}}{\mathrm{d}l}A_l(0),$$

after developing all Pochhammer symbols in the $\varepsilon \to 0$ form of (3) we arrive at the identity from [KR, Section 13]:

$$
\begin{aligned}
&-\sum_{l=0}^{n} \frac{\mathrm{d}}{\mathrm{d}l}\Big(\frac{n}{2}-l\Big)\binom{n}{l}^4\binom{n+l}{n}^2\binom{2n-l}{n}^2 \\
&\qquad = (-1)^n \sum_i \binom{n}{i}^2\binom{2n-i}{n}\sum_j \binom{n+j-i}{n}\binom{n}{j}^2\binom{n+j}{n}. \qquad (5)
\end{aligned}
$$

Clearly, the left-hand side of Andrews's formula is symmetric with respect to the group of parameters $b_1, c_1, b_2, c_2, b_3, c_3$. Therefore, setting as before $a = -n - 2\varepsilon$, $m = n$, and all the parameters of the group to be $-n - \varepsilon$, except the following two:

(a) $b_1 = c_1 = n - \varepsilon + 1$;
(b) $b_2 = c_2 = n - \varepsilon + 1$;
(c) $b_3 = c_3 = n - \varepsilon + 1$;
(d) $c_1 = c_2 = n - \varepsilon + 1$;
(e) $c_2 = c_3 = n - \varepsilon + 1$;
(f) $c_1 = c_3 = n - \varepsilon + 1$

(the last case corresponds to the above identity (5)), we arrive at the five more representations of the left-hand side of (5):

$$
\begin{aligned}
(-1)^n u_n &= -\sum_{l=0}^{n} \frac{\mathrm{d}}{\mathrm{d}l}\Big(\frac{n}{2}-l\Big)\binom{n}{l}^4\binom{n+l}{n}^2\binom{2n-l}{n}^2 \\
&= (-1)^n \sum_i (-1)^i\binom{3n+1}{i}\binom{2n-i}{n}^2\sum_j \binom{n+j-i}{n}\binom{n}{j}^2\binom{2n-j}{n} \\
&= (-1)^n \sum_i (-1)^i\binom{n+i}{n}^3\sum_j (-1)^j\binom{3n+1}{j-i}\binom{2n-j}{n}^3 \\
&= \sum_i \binom{n}{i}^2\binom{n+i}{n}\sum_j (-1)^j\binom{n+j-i}{n}\binom{n+j}{n}^2\binom{3n+1}{n-j} \\
&= (-1)^n \sum_i \binom{n}{i}\binom{n+i}{n}\binom{2n-i}{n}\sum_j \binom{n}{j-i}\binom{n}{j}\binom{2n-j}{n}^2 \\
&= (-1)^n \sum_i \binom{n}{i}\binom{n+i}{n}^2\sum_j \binom{n}{j-i}\binom{n}{j}\binom{n+j}{n}\binom{2n-j}{n}.
\end{aligned}
$$

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Moscow Lomonosov State University
Department of Mechanics and Mathematics
Vorobiovy Gory, GSP-2, Moscow 119992 RUSSIA
URL: http://wain.mi.ras.ru/index.html
E-mail address: wadim@ips.ras.ru, wzudilin@mi.uni-koeln.de